AVERAGING HIGHLY DISCONTINUOUS FUNCTIONS WITH UNDEFINED EXPECTED VALUES USING FAMILIES OF BOUNDED FUNCTIONS

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Abstract. Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function, where A and f are Borel. We want a unique, satisfying average of highly discontinuous f, taking finite values only. For instance, consider an everywhere surjective f, where its graph has zero Hausdorff measure in its dimension and a nowhere continuous f defined on the rationals. The problem is that the expected value of these examples of f, w.r.t. the Hausdorff measure in its dimension, is undefined. Thus, take any chosen family of bounded functions converging to f with the same satisfying and finite expected value, where the term "satisfying" is explained in the third paragraph.

The importance of this solution is that it solves the following problem: the set of all $f \in \mathbb{R}^A$ with a finite expected value, forms a shy "measure zero" subset of $\mathbb{R}^A$. This issue is solved since the set of all $f \in \mathbb{R}^A$, where there exists a family of bounded functions converging to f with a finite expected value, forms a prevalent "full measure" subset of $\mathbb{R}^A$. Despite this, the set of all $f \in \mathbb{R}^A$—where two or more families of bounded functions converging to f have different expected values—forms a prevalent subset of $\mathbb{R}^A$. Hence, we need a choice function which chooses a subset of all families of bounded functions converging to f with the same satisfying and finite expected value.

Notice, "satisfying" is explained in a leading question which uses rigorous versions of phrases in the former paragraph and the "measure" of the chosen families of each bounded function's graph involving partitioning each graph into equal measure sets and taking the following—a sample point from each partition, pathways of line segments between sample points, lengths of line segments in each pathway, removed lengths which are outliers, remaining lengths which are converted into a probability distribution, and the entropy of the distribution. In addition, we define a fixed rate of expansion versus the actual rate of expansion of a family of each bounded function's graph.

Keywords. Discontinuity, Hausdorff measure, Expected Value, Function Space, Prevalent and Shy Sets, Partitions, Samples, Euclidean Distance, Entropy, Choice Function

1. Intro

Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function, where A and f are Borel. We want a unique, satisfying average of highly discontinuous f, taking finite values only. For instance, consider an everywhere surjective f, where its graph has zero Hausdorff measure in its dimension (Section 2.1) and a nowhere continuous f defined on the rationals (Section 2.2).

The problem is the expected value of these examples of f, w.r.t. the Hausdorff measure in its dimension, is undefined (Section 2.3). In particular, the graph of an everywhere surjective[1] $f : \mathbb{R} \to \mathbb{R}$ could have Hausdorff dimension 2 with zero 2-d Hausdorff measure. Thereby, the function cannot be integrated w.r.t. the Hausdorff measure. Moreover, the expected value of a nowhere continuous $f : \mathbb{Q} \to \mathbb{R}$ can be any value between $\inf f$ and $\sup f$ depending on the enumeration of $\mathbb{Q}$ (Section 2.3).

To fix this dilemma, take the expected value of any family of bounded functions converging to f (Section 2.3.3); however, depending on the chosen family of bounded functions converging to f (Section 2.3.2), the expected value can be one of several values (Theorem 1). Hence, we define a leading question (Section 3.1) which chooses families of bounded functions with the same "satisfying" and finite expected value, where the term "satisfying" is explained rigorously.

We expand on the importance of the leading question with two problems (i.e., informal versions of Theorems 2 and 4):

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[1] The function $f : \mathbb{R} \to \mathbb{R}$ is everywhere surjective, when $f[(a, b)] = \mathbb{R}$ for all non-empty open intervals (a, b). The general definition can be found in Section 2.1.

1. If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where the expected value of f w.r.t. the Hausdorff measure in its dimension is finite, then F is shy (Section 2.4).
    - If $F \subset \mathbb{R}^A$ is shy [16], we say "**almost no**" element of $\mathbb{R}^A$ lies in F.
2. If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where two families of bounded functions converging to f have different expected values, then F is prevalent (Section 2.4).
    - If $F \subset \mathbb{R}^A$ is prevalent [16], we say "**almost all**" elements of $\mathbb{R}^A$ lies in F.

This means "almost no" $f \in \mathbb{R}^A$ have finite expected values and "almost all" $f \in \mathbb{R}^A$ have two or more families of bounded functions converging to f (Section 2.3.2) with different expected values. Hence, these problems need to be resolved by the leading question.

In Section 5, we clarify the leading question (Section 3.1) by applying the rigorous definitions of the question to specific examples (Section 5.2.1). We also define the "measure" (Section 5.3.1, Section 5.3.3) of the family of each bounded function's graph, such that the family of bounded functions converges to f (Section 2.3.2). This is crucial for defining a "satisfying" expected value.

The "measure" (Section 5.3.1, Section 5.3.3) is derived from examples of f with a discrete or countably infinite graph and generalized to f with non-discrete and uncountable graphs. Therefore, the "measure" is defined by the following:

1. Cover each graph with minimal, pairwise disjoint sets of equal ε Hausdorff measure (Section 5.3.1, step 1)
2. Take a sample point from each set in the cover (Section 5.3.1, step 2)
3. Take a "pathway of line segments" (Section 5.3.1, step 3a):
    (a) Take a line segment from the sample point $\mathbf{x}_0$ to a sample point (excluding $\mathbf{x}_0$) with the smallest Euclidean distance to $\mathbf{x}_0$ (i.e., when more than one point has the smallest Euclidean distance to $\mathbf{x}_0$, take either of those points). Call this point $\mathbf{x}_1$.
    (b) Take a line segment from the sample point $\mathbf{x}_1$ to a sample point (excluding $\mathbf{x}_0$ and $\mathbf{x}_1$) with the smallest Euclidean distance to $\mathbf{x}_1$ (i.e., when more than one point has the smallest Euclidean distance to $\mathbf{x}_1$, take either of those points). Call this point $\mathbf{x}_2$
    (c) Take a line segment from the sample point $\mathbf{x}_2$ to a sample point (excluding $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$) with the smallest Euclidean distance to $\mathbf{x}_2$ (i.e., when more than one point has the smallest Euclidean distance to $\mathbf{x}_2$, take either of those points). Call this point $\mathbf{x}_3$.
    (d) Repeat this process until the pathway intersects with every sample point once
4. Take the length of each line segment in the pathway and remove the outliers which are more than $C > 0$ times the interquartile range of the lengths of all line segments as $\varepsilon \to 0$ (Section 5.3.1, step 3b)
5. Multiply the remaining lengths by a constant to get a probability distribution (Section 5.3.1, step 3c)
6. Take the entropy of the distribution (Section 5.3.1, step 3d)
7. Take the maximum entropy w.r.t. all pathways (Section 5.3.1, step 3e)

Since the "measure" is extremely long and over-sophisticated, the definition needs to be simplified. To better understand this definition, consider these examples (Section 5.3.4-5.3.6).

Next, we define the fixed rate of expansion versus the actual rate of expansion of the family of each bounded function's graph (Section 5.4). The fixed rate of expansion is arbitrary and written as the function $E : \mathbb{R} \to \mathbb{R}$ (e.g., $E(r) = 1$). Meanwhile, the actual rate of expansion of a family of each bounded function's graph is a function of the "average" $(n + 1)$-dimensional Euclidean distance between every point in each bounded function's graph and the reference point $C \in \mathbb{R}^{n+1}$.

Combining "the measure" and the actual rate of expansion of each bounded function's graph, we get a general notion of the choice function in the leading question (Section 3.1). The choice function should pick a family of bounded functions converging to f which satisfy all the criteria in the leading question. Note, when all other criteria in the leading question (Section 3.1) are satisfied, then

1. the greater the rate of increase of the "measure" (Section 5.3.1, Section 5.3.3) of the chosen family of each bounded function's graph compared to that of other families of each bounded function's graph
2. the smaller the absolute difference between the $(n + 1)$-th coordinate of the reference point $C \in \mathbb{R}^{n+1}$ and the expected value of the chosen family of bounded functions converging to f (Section 2.3.2)
3. the smaller the absolute difference between the fixed rate of expansion and the actual rate of expansion of the chosen family of each bounded function's graph (Section 5.4),

the more likely the choice function, which answers the leading question, chooses the desired family. Hence, we take the expected value of the chosen family as the new expected value. (However, the choice function must actually take multiple families of bounded functions converging to f such that, for any function $f \in \mathbb{R}^A$, their expected values are the same "satisfying" and finite value.) We also want the new average of $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel, to exist for all f in a prevalent (Section 2.4) subset of $\mathbb{R}^A$.

Finally, we answer the leading question (Section 3.1) in Section 6. Since the answer is complicated, is likely incorrect, and the actual answer might not admit an unique expected value, it is best to keep refining the leading question (Section 3.1) rather than worrying about an immediate solution.

2. Formalizing the Intro

We want a unique, satisfying average of the functions in Sections 2.1 and 2.2, which takes finite values only. We explain a new method for averaging in later sections, starting with Section 2.3.1.

2.1. First special case of f.

If G is the graph of the function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel, we want an explicit f such that:

(1) The function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is everywhere surjective,

Let (A, T) be a standard topology. A function $f : A \to \mathbb{R}$ is *everywhere surjective* from A to $\mathbb{R}$, if $f[V] = \mathbb{R}$ for every $V \in \mathrm{T}$.

(2) If $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension and $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra, then $\mathcal{H}^{\dim_{\mathrm{H}}(G)}(G) = 0$.

2.1.1. *Potential Example.* If $A = \mathbb{R}$, using this post [5]:

> Consider a Cantor set $\mathcal{C} \subseteq [0,1]$ with Hausdorff dimension 0 [6]. Now consider a countable disjoint union $\cup_{m\in\mathbb{N}} \mathcal{C}_m$ such that each $\mathcal{C}_m$ is the image of $\mathcal{C}$ by some affine map and every open set $O \subseteq [0,1]$ contains $\mathcal{C}_m$ for some m. Such a countable collection can be obtained by e.g. at letting $\mathcal{C}_m$ be contained in the biggest connected component of $[0,1] \setminus (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{m-1})$ (with the center of $\mathcal{C}_m$ being the middle point of the component).
>
> Note that $\cup_m \mathcal{C}_m$ has Hausdorff dimension 0, so $(\cup_m \mathcal{C}_m) \times [0,1] \subseteq \mathbb{R}^2$ has Hausdorff dimension one [4].
>
> Now, let $g : [0,1] \to \mathbb{R}$ such that $g|_{\mathcal{C}_m}$ is a bijection $\mathcal{C}_m \to \mathbb{R}$ for all m (all of them can be constructed from a single bijection $\mathcal{C} \to \mathbb{R}$, which can be obtained without choice, although it may be ugly to define) and outside $\cup_m \mathcal{C}_m$ let g be defined by $g(x) = h(x)$, where $h : [0,1] \to \mathbb{R}$ has a graph with Hausdorff dimension 2 [17] (this doesn't require choice either).
>
> Then, the function g has a graph with Hausdorff dimension 2 and is everywhere surjective, but its graph has Lebesgue measure 0 because it is a graph (so it admits uncountably many disjoint vertical translates).
>
> Note, we can make the construction with union of $\mathcal{C}_m$ rather explicit as follows. Split the binary expansion of x as strings of size with a power of two, say $x = 0.1101000010\ldots$ becomes $(s_0, s_1, s_2, \ldots) = (1, 10, 1000, \ldots)$. If this family eventually contains only strings of the form $0 \cdots 0$ or $1 \cdots 1$, say after s_k, then send it to $y = \sum_{i>0} \epsilon_i 2^{-i}$, where $s_{k+i} = \epsilon_i \cdots \epsilon_i$. Otherwise, send it to the explicit continuous function h given by the linked article [17]. This will give you something from $[0,1) \to [0,1)$
>
> Finally, compose an explicit (reasonable) bijection from $[0,1)$ to $\mathbb{R}$. In your case, the construction can be easily adapted so that the $[0,1]$ or $[0,1)$ target space is actually $(0,1)$, then compose with $t \mapsto (1-2x)/(x^2-x)$.

In case we cannot obtain a unique, satisfying, and finite average (Section 3.1) from this example in Section 2.1.1, consider the following:

2.2. Second special case of f.

Suppose, we define $A = \mathbb{Q}$, where $f : A \to \mathbb{R}$ is a function:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \end{cases} \tag{1}$$

In the next section, we state the purpose of Section 2.1 and Section 2.2.

2.3. **Motivation for Section 2.1 and 2.2.** Suppose,

1. $\dim_{\text{H}}(\cdot)$ is the Hausdorff dimension
2. $\mathcal{H}^{\dim_{\text{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra
3. the integral is defined, w.r.t. the Hausdorff measure in its dimension.

the expected value of $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel, w.r.t. the Hausdorff measure in its dimension is the following:

$$\mathbb{E}[f] = \frac{1}{\mathcal{H}^{\dim_{\text{H}}(A)}(A)} \int_A f \, d\mathcal{H}^{\dim_{\text{H}}(A)} \tag{2}$$

Then, using Section 2.1.1, the integral of f w.r.t. the Hausdorff measure in its dimension is undefined: i.e., the graph of f has Hausdorff dimension 2 with zero 2-d Hausdorff measure. Hence, $\mathbb{E}[f]$ is not defined.

Moreover, observe in Section 2.2, f is nowhere continuous and defined on a countably infinite set, which means depending on the enumeration of $A = \mathbb{Q}$ or the sequence $\{a_r\}_{r=1}^{\infty}$, the expected value of f (when it exists) is:

$$\mathbb{E}[f] = \lim_{t \to \infty} \frac{f(a_1) + f(a_2) + \cdots + f(a_t)}{t} \tag{3}$$

which is any number from $\inf f$ to $\sup f$. Thus, we need a specific enumeration which gives a unique, satisfying, and finite expected value, generalizing this process to nowhere continuous functions defined on **uncountable** domains.

Therefore, we want the "expected value of chosen families of bounded functions converging to f with the same satisfying and finite expected value" which is described rigorously in later sections; however, consider the following definitions beginning with Section 2.3.1:

2.3.1. *Definition of Family of Functions and Sets.* Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function, where A and f are Borel.

Let $\dim_{\text{H}}(\cdot)$ be the Hausdorff dimension, where $\mathcal{H}^{\dim_{\text{H}}(\cdot)}(\cdot)$ be the Hausdorff dimension measure in its dimension on the Borel σ-algebra. If $\beta(A) \subseteq A$ is the index set and the cardinality is $|\cdot|$, then suppose $|\beta(A)| = |A|$. Thus, for every $a, b \in \mathbb{R}$ where:

$$\frac{\mathcal{H}^{\dim_{\text{H}}(A)}(\mathcal{A}(A) \cap (a,b))}{\mathcal{H}^{\dim_{\text{H}}(A)}(A \cap (a,b))} = \inf_{\beta(A) \subseteq A} \left(\frac{\mathcal{H}^{\dim_{\text{H}}(A)}(\beta(A) \cap (a,b))}{\mathcal{H}^{\dim_{\text{H}}(A)}(A \cap (a,b))} \right),$$

then for each $r \in \mathcal{A}(A)$, there is a corresponding set A_r and a corresponding function $f_r : A_r \to \mathbb{R}$ such that the indexed family of sets is $\{A_r : r \in \mathcal{A}(A)\}$ and the indexed family of functions is $\{f_r : r \in \mathcal{A}(A)\}$.

Note, when A is countably infinite, then $\mathcal{A}(A) = \mathbb{N}$ and when $A = \mathbb{R}^n$, $\mathcal{A}(A) = \{(\alpha_1, \cdots, \alpha_n) \in A : q \in \{1, \cdots, n\}, \alpha_q > 0\}$.

2.3.2. *Definition of Families of Functions Converging to f.* Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function, where A and f are Borel.

The family of functions $\{f_r : r \in \mathcal{A}(A)\}$ (Section 2.3.1), where $\{A_r : r \in \mathcal{A}(A)\}$ is a family of sets and $f_r : A_r \to \mathbb{R}$ is a function, converges to f when:

> For any $(x_1, \cdots, x_n) \in A$, there exists an indexed family $\bar{x}_r \in A_r$ *s.t.* $\bar{x}_r \to (x_1, \cdot\cdot\cdot, x_n)$ and $f_r(\bar{x}_r) \to f(x_1, \cdot\cdot\cdot, x_n)$.

This is equivalent to:

$$(f_r, A_r) \to (f, A)$$

2.3.3. *Expected Value of Families of Functions Converging to f.* Hence, suppose:

- $(f_r, A_r) \to (f, A)$ (Section 2.3.2)
- $||\cdot||$ is the absolute value
- $\dim_{\text{H}}(\cdot)$ is the Hausdorff dimension
- $\mathcal{H}^{\dim_{\text{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra
- the integral is defined, w.r.t. the Hausdorff measure in its dimension

The expected value of $\{f_r : r \in \mathcal{A}(A)\}$ is a real number $\mathbb{E}[f_r]$, when the following is true:

$$\forall(\epsilon > 0)\exists(N \in \mathcal{A}(A))\forall(r \in \mathcal{A}(A))\left(r \geq N \Rightarrow \left\|\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(A_r)}\int_{A_r} f_r \, d\mathcal{H}^{\dim_{\mathrm{H}}(A_r)} - \mathbb{E}[f_r]\right\| < \epsilon\right) \tag{4}$$

when no such $\mathbb{E}[f_r]$ exists, $\mathbb{E}[f_r]$ is infinite or undefined. (If the graph of f has zero Hausdorff measure in its dimension, replace $\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33].)

2.3.4. *The Set of All Bounded Functions/Sets.* Let $n \in \mathbb{N}$ and suppose the function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel. Let $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension, where $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ be the Hausdorff measure in its dimension on the Borel σ-algebra. Then, define the following:
Let $\mathbf{Q}$ be a set.

1. $\mathbf{B}(\mathbf{Q})$ is the set of all bounded Borel sets $\mathbf{X} \subseteq \mathbf{Q}$ where $0 < \mathcal{H}^{\dim_{\mathrm{H}}(\mathbf{X})}(\mathbf{X}) < +\infty$
2. $\mathfrak{B}(\mathbf{Q})$ is the set of all bounded Borel functions with domain $\mathbf{Q}$ where $0 < \mathcal{H}^{\dim_{\mathrm{H}}(\mathbf{Q})}(\mathbf{Q}) < +\infty$

For example, $\mathbf{B}(\mathbb{R}^n)$ is the set of all bounded Borel subsets of $\mathbb{R}^n$ with a positive and finite Hausdorff measure in the dimension of X and $\mathfrak{B}([1,2])$ is the set of all bounded Borel functions on $[1,2]$ since $\mathcal{H}^1([1,2]) = 1$ and $0 < \mathcal{H}^1([1,2]) < +\infty$. Note, however:

Theorem 1. *For all $r, v \in \mathcal{A}(A)$ (Section 2.3.1), suppose $A_r, B_v \in \mathbf{B}(\mathbb{R}^n)$, where $f_r \in \mathfrak{B}(A_r)$ and $g_v \in \mathfrak{B}(B_v)$ (Section 2.3.4). There exists a $f \in \mathbb{R}^A$, where $(f_r, A_r), (g_v, B_v) \to (f, A)$ (Section 2.3.2) and $\mathbb{E}[f_r] \neq \mathbb{E}[g_v]$ (Section 2.3.3).*

For instance, the expected values of the families of bounded functions converging to f in Section 2.1 and 2.2 satisfy Theorem 1. For simplicity, we illustrate this with Section 2.2.

2.3.5. *Example Illustrating Theorem 1.* For the second case of Borel $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ (Section 2.2), where $A = \mathbb{Q}$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \end{cases} \tag{5}$$

suppose:

$$\{A_r : r \in \mathcal{A}(A) := \mathbb{N}\} = (\{c/r! : c \in \mathbb{Z}, -r \cdot r! \leq c \leq r \cdot r!\})_{r \in \mathbb{N}}$$

and

$$\{B_v : v \in \mathcal{A}(A) := \mathbb{N}\} = (\{c/d : c \in \mathbb{Z}, d \in \mathbb{N}, d \leq v, -d \cdot v \leq c \leq d \cdot v\})_{v \in \mathbb{N}}$$

where for $f_r : A_r \to \mathbb{R}$,

$$f_r(x) = f(x) \text{ for all } x \in A_r \tag{6}$$

and for $g_v : B_v \to \mathbb{R}$

$$g_v(x) = f(x) \text{ for all } x \in B_v \tag{7}$$

Note, for all $r, v \in \mathbb{N}$:

- $\sup(A_r) = r$
- $\inf(A_r) = -r$
- $\sup(B_v) = v$
- $\inf(B_v) = -v$
- Since f is bounded, f_r and g_v are bounded

Hence, $A_r, B_v \in \mathbf{B}(\mathbb{R}^n)$, where $f_r \in \mathfrak{B}(A_r)$ and $g_v \in \mathfrak{B}(B_v)$ (Section 2.3.4). Also, the set-theoretic limit of $\{A_r : r \in \mathbb{N}\}$ and $\{B_v : v \in \mathbb{N}\}$ is $A = \mathbb{Q}$: i.e.,

$$\limsup_{r \to \infty} A_r = \bigcap_{r \geq 1} \bigcup_{q \geq r} A_q$$
$$\liminf_{r \to \infty} A_r = \bigcup_{r \geq 1} \bigcap_{q \geq r} A_q$$

where:

$$\limsup_{r \to \infty} A_r = \liminf_{r \to \infty} A_r = A = \mathbb{Q}$$
$$\limsup_{v \to \infty} B_v = \liminf_{v \to \infty} B_v = A = \mathbb{Q}$$

(We do not not know how to prove the set-theoretic limits; however, a mathematician specializing in analysis should be able to confirm.)

Therefore, $(f_r, A_r), (g_v, B_v) \to (f, A)$ (Theorem 1).

Now, suppose we want to average $\{f_r : r \in \mathcal{A}(A) := \mathbb{N}\}$ and $\{g_v : v \in \mathcal{A}(A) := \mathbb{N}\}$, which we denote $\mathbb{E}[f_r]$ and $\mathbb{E}[g_v]$. Note, this is the same as computing the following (i.e., the cardinality is $|\cdot|$ and the absolute value is $||\cdot||$):

$$\begin{aligned} &\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left|\left| \frac{1}{|A_r|} \int_{A_r} f \, d\mathcal{H}^0 - \mathbb{E}[f_r] \right|\right| < \epsilon \right) \implies \\ &\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left|\left| \frac{1}{|A_r|} \sum_{x \in A_r} f(x) - \mathbb{E}[f_r] \right|\right| < \epsilon \right) \end{aligned} \tag{8}$$

$$\begin{aligned} &\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(v \in \mathbb{N}) \left(v \geq N \Rightarrow \left|\left| \frac{1}{|B_v|} \int_{B_v} f \, d\mathcal{H}^0 - \mathbb{E}[g_v] \right|\right| < \epsilon \right) \implies \\ &\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(v \in \mathbb{N}) \left(v \geq N \Rightarrow \left|\left| \frac{1}{|B_v|} \sum_{x \in B_v} f(x) - \mathbb{E}[g_v] \right|\right| < \epsilon \right) \end{aligned} \tag{9}$$

Thus, if we assume $\mathbb{E}[f_r] = 1$ in Equation 8, using [12]:

> The sum $\sum_{x \in A^\star} f(x)$ counts the number of fractions with an even denominator and an odd numerator in set $A^\star$, after canceling all possible factors of 2 in the fraction. Let us consider the first case. We can write:
>
> $$\left|\left| 1 - |A_r|^{-1} \sum_{x \in A_r} f(x) \right|\right| = \left(|A_r| - \sum_{x \in A_r} f(x) \right) / |A_r| = H(r)/|A_r|$$
>
> where $H(r)$ counts the fractions $x = c/r!$ in A_r that are not counted in $\sum_{x \in A^\star} f(x)$, i.e., for which $f(x) = 0$. This is the case when the denominator is odd after the cancellation of the factors of 2, i.e., when the numerator c has a number of factors of 2 greater than or equal to that of $r!$, which we will denote by $V(r) := v_2(r!)$ a.k.a the 2-valuation of $r!$, oeis:A11371$(r) = r - O(\ln(r))$ [14]. That means, c must be a multiple of $2^{V(r)}$. The number of such c with $-r \cdot r! \leq c \leq r \cdot r$ is simply the length of that interval, equal to $|A_r| = 2r(r!) + 1$, divided by $2^{V(r)}$. Thus,
>
> $$\left|\left| 1 - |A_r|^{-1} \sum_{x \in A_r} f(x) \right|\right| = [|A_r|/2^{V(r)}]/|A_r| \sim 1/2^{V(r)} = 1/2^{n - O(\log n)}$$
>
> This obviously tends to zero, proving $\mathbb{E}[f_r] = 1$

Last, we need to show $\mathbb{E}[g_v] = 1/3$ in Equation 9, where $1 = \mathbb{E}[f_r] \neq \mathbb{E}[g_v] = 1/3$. This proves Theorem 1.

> Concerning the second case [12], it is again simpler to consider the complementary set of $x \in B_v$ such that the denominator is odd when all possible factors of 2 are canceled. We can see that for $v = 2p - 1$, and these obviously include all those we had for smaller v. The "new" elements in B_v with $v = 2p - 1$ are those that have the denominator $d = 2p - 1$ when written in lowest terms. Their number is equal to the number of $\kappa < d$, $\gcd(\kappa, d) = 1$, which is given by Euler's ϕ function. Since we also consider negative fractions, we have to multiply this by 2. Including $x = 0$, we have $G(v) = |\{x \in B_v | f(x) = 0\}| = 1 + 2\sum_{0 \leq \kappa \leq v/2} \phi(2\kappa + 1)$. There is no simple explicit expression for this (cf. oeis:A99957 [15]), but we know that $G(v) = 1 + 2 \cdot \text{A99957}(v/2) \sim 2 \cdot 8(v/2)^2/\pi^2 = 4v^2/\pi^2$ [15]. On the other hand, the total number of all elements of B_v is $|B_v| = 1 + 2\sum_{1 \leq \kappa \leq v} \phi(\kappa)$, since each time we increase v by 1, we have the additional fractions with the new denominator $d = v$ and the numerators are coprime with d, again with the sign + or −. From oeis:A002088

[13] we know that $\sum_{1\leq\kappa\leq v}\phi(\kappa) = 3v^2/\pi^2 + O(v\log v)$, so $|B_v| \sim 6v^2/\pi^2$, which finally gives $|B_v|^{-1}\sum_{x\in B_v} f(x) = (|B_v| - G(v))/|B_v| \sim (6-4)/6 = 1/3$ as desired.

Hence, $1/3 = \mathbb{E}[g_v] \neq \mathbb{E}[f_r] = 1$, proving Theorem 1. Thus, consider:

2.4. **Definition of Prevalent and Shy Sets.** Let X be a completely metrizable topological space. A Borel set $E \subset X$ is said to be **prevalent** if there exists a Borel measure μ on X such that:

1. $0 < \mu(C) < \infty$ for some compact subset C of X, and
2. the set $E + x$ has full μ-measure (that is, the complement of $E + x$ has measure zero) for all $x \in X$.

More generally, a subset F of X is prevalent if F contains a prevalent Borel Set.
Moreover:

- The complement of a prevelant set is a **shy** set.

Hence:

- If $F \subset X$ is *prevelant*, we say "**almost every**" element of X lies in F.
- If $F \subset X$ is *shy*, we say "**almost no**" element of X lies in F.

To learn more, see [16].

2.5. **Secondary Motivation For A New Average.** Suppose,

1. $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension
2. $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra
3. the integral is defined, w.r.t. the Hausdorff measure in its dimension.

the expected value of $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel, w.r.t. the Hausdorff measure in its dimension is the following:

$$\mathbb{E}[f] = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A)}(A)}\int_A f\, d\mathcal{H}^{\dim_{\mathrm{H}}(A)} \tag{10}$$

Hence, consider the following:

Theorem 2. *If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where $\mathbb{E}[f]$ is finite, then F is shy (Section 2.4).*

Issue with Theorem 2. "Almost no" $f \in \mathbb{R}^A$ have finite expected values, which is important since finite averages are more useful than infinite or undefined averages.

Proof of Theorem 2. We follow the argument presented in Example 3.6 of this paper [16], take $X := L^0(A)$ (measurable functions over A), let P denote the one-dimensional subspace of A consisting of constant functions (assuming the Lebesgue measure on A) and let $F := L^0(A) \setminus L^1(A)$ (measurable functions over A without finite integral). Let λ_P denote the Lebesgue measure over P, for any fixed $f \in F$:

$$\lambda_P\left(\left\{\alpha\in\mathbb{R}\middle| \int_A (f+\alpha)d\mu < \infty\right\}\right) = 0$$

Meaning P is a one-dimensional, so f is a 1-prevalent set □

We solve the issue in Theorem 2 with Note 3.

Note 3 (**Solving The Issue In Theorem 2**). *For all $r \in \mathcal{A}(A)$ (Section 2.3.1), suppose that $A_r \in \mathbf{B}(\mathbb{R}^n)$ and $f_r \in \mathfrak{B}(A_r)$ (Section 2.3.4). If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where there exists $A_r \in \mathbf{B}(\mathbb{R}^n)$ and $f_r \in \mathfrak{B}(A_r)$ such that $(f_r, A_r) \to (f, A)$ (Section 2.3.2) and $\mathbb{E}[f_r]$ is finite (Section 2.3.3), then F should be prevalent (Section 2.4) or neither prevalent nor shy (Section 2.4).*

Theorem 4. *For all $r, v \in \mathcal{A}(A)$ (Section 2.3.1), suppose $A_r, B_v \in \mathbf{B}(\mathbb{R}^n)$, where $f_r \in \mathfrak{B}(A_r)$ and $g_v \in \mathfrak{B}(B_v)$ (Section 2.3.4). When $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where $(f_r, A_r), (g_v, B_v) \to (f, A)$ (Section 2.3.2) and $\mathbb{E}[f_r] \neq \mathbb{E}[g_v]$ (Section 2.3.3), then F is prevalent (Section 2.4).*

Issue with Theorem 4. For “almost all” $f \in \mathbb{R}^A$, depending on the chosen family of bounded functions $\{f_r : r \in \mathcal{A}(A)\}$ where $(f_r, A_r) \to (f, A)$ (Section 2.3.2), $\mathbb{E}[f_r]$ can be more than one value. Hence, $\mathbb{E}[f_r]$ is non-unique.

Proof of Theorem 4. For all $r, v \in \mathcal{A}(A)$ (Section 2.3.1), suppose $A_r, B_v \in \mathbf{B}(\mathbb{R}^n)$ where $f_r \in \mathfrak{B}(A_r)$ and $g_v \in \mathfrak{B}(B_v)$ (Section 2.3.4). Therefore, suppose $U \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$ whose lines of symmetry intersect at one point, where if $(f_r, A_r), (g_v, B_v) \to (f, A)$ (Section 2.3.2), then $\mathbb{E}[f_r] = \mathbb{E}[g_v]$ (Section 2.3.3). In addition, $U' \subset \mathbb{R}^A$ is the set of symmetric $f \in \mathbb{R}^A$ which clearly forms a shy subset of $\mathbb{R}^A$. Since $U \subset U'$, we have proven that U is also shy (i.e., a subset of a shy set is also shy). Since the complement of the shy set U is prevalent, $F = \mathbb{R}^A \setminus U$ is prevalent, such that for all $f \in F$, $(f_r, A_r), (g_v, A_v) \to (f, A)$ and $\mathbb{E}[f_r] \neq \mathbb{E}[g_v]$. If this is correct, we have proven Theorem 4. ☐

We solve the issue in Theorem 4 with Note 5.

Note 5 (**Solving The Issue In Theorem 4**). *Suppose* $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$ *is an arbitrary set, where for all* $r \in \mathcal{A}(A)$ *(Section 2.3.1),* $A_r \in \mathcal{B}$ *and* $\mathscr{B} \subset \mathfrak{B}(A_r)$*, such that* $f_r \in \mathscr{B}$ *(Section 2.3.4). If* $F \subset \mathbb{R}^A$ *is the set of all* $f \in \mathbb{R}^A$*, where* $(f_r, A_r) \to (f, A)$ *(Section 2.3.2) and* $\mathbb{E}[f_r]$ *is unique (Section 2.3.3), then* F *should be prevelant (Section 2.4).*

Since Theorems 2 and 4 are true, we need to solve both Theorems *at once* by combining Notes 3 and 5. (See Section 2.5.1.)

2.5.1. Approach.

> Suppose $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$ is an arbitrary set, where for all $r \in \mathcal{A}(A)$ (Section 2.3.1), $A_r \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r)$ such that $f_r \in \mathscr{B}$ (Section 2.3.4). If $F \subset \mathbb{R}^A$ is the collection of all $f \in \mathbb{R}^A$, where $(f_r, A_r) \to (f, A)$ (Section 2.3.2) and $\mathbb{E}[f_r]$ is unique (Section 2.3.3), satisfying (Section 3) and finite, then F should be:
>
> (1) a prevalent (Section 2.4) subset of $\mathbb{R}^A$
> (2) If not prevalent (Section 2.4) then neither prevalent (Section 2.4) nor shy (Section 2.4) subset of $\mathbb{R}^A$.

3. Attempt to Define “Satisfying” in The Approach of Section 2.5.1

3.1. **Leading Question.** To define *satisfying* in the blockquote of Section 2.5.1, we ask the **leading question**...

> Suppose, for all $r, v \in \mathcal{A}(A)$ (Section 2.3.1), $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$ is an arbitrary set (Section 2.3.4), where $A_r^{\star} \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r^{\star})$ (Section 2.3.4) such that:
>
> (A) $f_r^{\star} \in \mathscr{B}$ (Section 2.3.4)
> (B) $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$ and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star}) \cup (\mathfrak{B}(A_r^{\star}) \setminus \mathscr{B})$ (Section 2.3.4)
> (C) $\{G_r^{\star} : r \in \mathcal{A}(A)\} = \{\mathrm{graph}(f_r^{\star}) : r \in \mathcal{A}(A)\}$ is the family of the graph of each $f_r^{\star}$ (Section 2.3.2)
> (D) C is a reference point in $\mathbb{R}^{n+1}$ (e.g., the origin)
> (E) $E : \mathcal{A}(A) \to \mathbb{R}$ is a function and the fixed rate of expansion: e.g., $E(r) = 1$ (Section 3.1.C, Section 3.1.D)
> (F) $\mathcal{E}(C, G_r^{\star})$ is the actual rate of expansion of $\{G_r^{\star} : r \in \mathcal{A}(A)\}$ w.r.t. a reference point C (Section 3.1.C, Section 3.1.D, Section 5.4)
>
> Does there exist a choice function which chooses a set $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, where for all $r \in \mathcal{A}(A)$, $A_r^{\star} \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r^{\star})$ such that when $f_r^{\star} \in \mathscr{B}$:
>
> (1) $(f_r^{\star}, A_r^{\star}) \to (f, A)$ (Section 2.3.2)
> (2) For all $v \in \mathcal{A}(A)$, where for each $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$ and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star}) \cup (\mathfrak{B}(A_r^{\star}) \setminus \mathscr{B})$, when $(f_v^{\star\star}, A_v^{\star\star}) \to (f, A)$, the “measure” (Section 5.3.1, Section 5.3.3) of $\{G_r^{\star} : r \in \mathcal{A}(A)\} = \{\mathrm{graph}(f_r^{\star}) : r \in \mathcal{A}(A)\}$ (Section 3.1.C) must increase at a rate linear or superlinear to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\} = \{\mathrm{graph}(f_v^{\star\star}) : v \in \mathcal{A}(A)\}$ (Section 3.1.C)
> (3) $\mathbb{E}[f_r^{\star}]$ is unique and finite (Section 2.3.3)

(4) For each $A_r^\star \in \mathcal{B}$ and $f_r^\star \in \mathscr{B}$ satisfying (1), (2) and (3), when f is unbounded (i.e, skip (4) when f is bounded), then for every set $\mathcal{B}' \subset \mathbf{B}(\mathbb{R}^n)$ and for all $s \in \mathbb{N}$, where for each $A_s^{\star\star\star} \in \mathcal{B}'$ and for every set $\mathscr{B}' \subset \mathfrak{B}(A_s^{\star\star\star})$, when $\star \mapsto \star\star\star$, $r \mapsto s$, $\mathcal{B} \mapsto \mathcal{B}'$, and $\mathscr{B} \mapsto \mathscr{B}'$ in (1), (2) and (3), then when $f_s^{\star\star\star} \in \mathscr{B}'$ satisfies (1), (2) and (3):
   - If the absolute value is $||\cdot||$ and the $(n+1)$-th coordinate of C (Section 3.1.D) is x_{n+1}, $||\mathbb{E}[f_r^\star] - x_{n+1}|| \le ||\mathbb{E}[f_s^{\star\star\star}] - x_{n+1}||$ (Section 2.3.2, Section 2.3.3)
   - If $r \in \mathcal{A}(A)$, then for all linear $s_1 : \mathcal{A}(A) \to \mathcal{A}(A)$, where $s = s_1(r)$ and the Big-O notation is $\mathcal{O}$, there exists a function $K : \mathbb{R} \to \mathbb{R}$, where the absolute value is $||\cdot||$ and (Section 3.1.E-F):

$$\begin{aligned} ||\mathcal{E}(C, G_r^\star) - E(r)|| =& \mathcal{O}(K(||\mathcal{E}(C, G_s^{\star\star\star}) - E(s)||)) \\ =& \mathcal{O}(K(||\mathcal{E}(C, G_{s_1(r)}^{\star\star\star}) - E(s_1(r))||)) \end{aligned}$$

   such that:

$$0 \le \lim_{x \to +\infty} K(x)/x < +\infty$$

   In simpler terms, "the rate of divergence" of $||\mathcal{E}(C, G_r^\star) - E(r)||$ (Section 3.1.E-F) is *less than or equal* to "the rate of divergence" of $||\mathcal{E}(C, G_s^{\star\star\star}) - E(s)||$ (Section 3.1.E-F).

(5) When set $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where a choice function chooses a collection $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, where $A_r^\star \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r^\star)$ such that $f_r^\star \in \mathscr{B}$ satisfies (1), (2), (3) and (4), then F should be:
   (a) a prevelant (Section 2.4) subset of $\mathbb{R}^A$
   (b) If not (a), then neither a prevalent (Section 2.4) nor shy (Section 2.4) subset of $\mathbb{R}^A$

(6) Out of all choice functions which satisfy (1), (2), (3), (4) and (5), we choose the one with the simplest form, meaning for each choice function fully expanded, we take the one with the fewest variables/numbers?

Note 6 (Checking The Validity of The Leading Question). *Unless the choice function chooses all $f_r^\star \in \mathscr{B}$ (Section 3.1.A) which are equivalent to each other (Note 8, page 25), $\mathbb{E}[f_r^\star]$ (Section 3.1 crit. 3) might not be unique nor satisfying enough to answer the approach of Section 2.5.1. Hence, adjustments are possible by changing the criteria or adding new criteria to Section 3.1.*

(In case this is unclear, see Section 5.)

4. QUESTION REGARDING MY WORK

Most researchers do not have time to address everything in this paper, hence we ask the following:

> *Is there a research paper which already solves the ideas in this paper? (Non-published papers, such as* [10]*, do not count.)*

Note, these papers might be useful [7, 8, 2, 3].

5. CLARIFYING SECTION 3

While reading Section 5, consider the following:

> Is there a simpler version of the definitions below?

5.1. Example of families of Bounded Functions Converging to f (Section 2.3.2).

Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function, where A and f are Borel.

The family of functions $\{f_r : r \in \mathcal{A}(A)\}$ (Section 2.3.1), where $\{A_r : r \in \mathcal{A}(A)\}$ is a family of bounded sets and $f_r : A_r \to \mathbb{R}$ is a bounded function, converges to f when:

> For any $(x_1, \cdots, x_n) \in A$, there exists an indexed family $\bar{x}_r \in A_r$ *s.t.* $\bar{x}_r \to (x_1, \cdot\cdot\cdot, x_n)$ and $f_r(\bar{x}_r) \to f(x_1, \cdot\cdot\cdot, x_n)$.

This is equivalent to:

$$(f_r, A_r) \to (f, A)$$

Example 0.1 (Example of Section 5.1). *If $A = \mathbb{R}$ and $f : A \to \mathbb{R}$, where $f(x) = 1/x$, then an example of $\{f_r : r \in \mathcal{A}(A) := \mathbb{R}^+\}$ (Section 2.3.1), such that $f_r : A_r \to \mathbb{R}$ is:*

(1) $\{A_r : r \in \mathcal{A}(A) := \mathbb{R}^+\} = \{[-r,-1/r] \cup [1/r,r] : r \in \mathbb{R}^+\}$
(2) $f_r(x) = 1/x$ *for* $x \in A_r$

Example 0.2 (**More Complex Example**). *If* $A = \mathbb{R}$ *and* $f : A \to \mathbb{R}$, *where* $f(x) = x$, *then an example of* $\{f_r : r \in \mathcal{A}(A) := \mathbb{R}^+\}$ *(Section 2.3.1), such that* $f_r : A_r \to \mathbb{R}$ *is:*

(1) $\{A_r : r \in \mathcal{A}(A) := \mathbb{R}^+\} = \{[-r,r] : r \in \mathbb{R}^+\}$
(2) $f_r(x) = x + (1/r)\sin(x)$ *for* $x \in A_r$

5.2. **Expected Value of a Bounded family of Functions (Section 2.3.3).** Hence, suppose:

- $(f_r, A_r) \to (f, A)$ (Section 5.1)
- $||\cdot||$ is the absolute value
- $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension
- $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra
- the integral is defined, w.r.t. the Hausdorff measure in its dimension

The expected value of $\{f_r : r \in \mathcal{A}(A)\}$ (Section 2.3.1) is a real number $\mathbb{E}[f_r]$, when the following is true:

$$\forall(\epsilon > 0)\exists(N \in \mathcal{A}(A))\forall(r \in \mathcal{A}(A))\left(r \geq N \Rightarrow \left|\left|\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(A_r)}\int_{A_r} f_r \, d\mathcal{H}^{\dim_{\mathrm{H}}(A_r)} - \mathbb{E}[f_r]\right|\right| < \epsilon\right) \tag{11}$$

when no such $\mathbb{E}[f_r]$ exists, $\mathbb{E}[f_r]$ is infinite or undefined. (If the graph of f has zero Hausdorff measure in its dimension, replace $\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33].)

5.2.1. *Example.* Using Example 0.1, when $A = \mathbb{R}$, $f : A \to \mathbb{R}$, and $f(x) = 1/x$ where:

(1) $\{A_r : r \in \mathcal{A}(A) := \mathbb{R}^+\} = \{[-r,-1/r] \cup [1/r,r] : r \in \mathbb{R}^+\}$
(2) $f_r(x) = 1/x$ for $x \in A_r$

If we assume $\mathbb{E}[f_r] = 0$:

$$\forall(\epsilon > 0)\exists(N \in \mathbb{R})\forall(r \in \mathbb{R})\left(r \geq N \Rightarrow \left|\left|\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(A_r)}\int_{A_r} f_r \, d\mathcal{H}^{\dim_{\mathrm{H}}(A_r)} - \mathbb{E}[f_r]\right|\right| < \epsilon\right) \tag{12}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{R})\forall(r \in \mathbb{R})\Big(r \geq N \Rightarrow \tag{13}$$

$$\left|\left|\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}([-r,-1/r]\cup[1/r,r])}([-r,-1/r] \cup [1/r,r])}\int_{[-r,-1/r]\cup[1/r,r]} 1/x \, d\mathcal{H}^{\dim_{\mathrm{H}}([-r,-1/r]\cup[1/r,r])} - 0\right|\right| < \epsilon\Big) \tag{14}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{R})\forall(r \in \mathbb{R})\left(r \geq N \Rightarrow \left|\left|\frac{1}{\mathcal{H}^{1}([-r,-1/r] \cup [1/r,r])}\int_{[-r,-1/r]\cup[1/r,r]} 1/x \, d\mathcal{H}^{1}\right|\right| < \epsilon\right) \tag{15}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{R})\forall(r \in \mathbb{R})\left(r \geq N \Rightarrow \left|\left|\frac{1}{(-1/r-(-r))+(r-1/r)}\left(\int_{-r}^{-1/r} 1/x \, dx + \int_{1/r}^{r} 1/x \, dx\right)\right|\right| < \epsilon\right) \tag{16}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{R})\forall(r \in \mathbb{R})\left(r \geq N \Rightarrow \left|\left|\frac{1}{(r-1/r)+(-1/r+r)}\left(\ln(||x||) + C\Big|_{-r}^{-1/r} + \ln(||x||) + C\Big|_{1/r}^{r}\right)\right|\right| < \epsilon\right) \tag{17}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{R})\forall(r \in \mathbb{R})\left(r \geq N \Rightarrow \left|\left|\frac{1}{(r-1/r)+(-1/r+r)}(\ln(||-r||) - \ln(||-1/r||) + \ln(||r||) - \ln(||1/r||))\right|\right| < \epsilon\right) \tag{18}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{R})\forall(r \in \mathbb{R})\left(r \geq N \Rightarrow \left|\left|\frac{1}{2r-2/r}\cdot 4\ln(r)\right|\right| < \epsilon\right) \tag{19}$$

To prove Equation 19 is true, recall:

$$r \ll e^{r/2},\ e^{1/r} \ll e^{r} \tag{20}$$

$$r \ll e^{r/2},\ e^{1/(2r)} \ll e^{r/2} \tag{21}$$

$$re^{1/(2r)} \ll e^{r/2} \tag{22}$$

$$r \ll e^{r/2}/e^{1/(2r)} \tag{23}$$

$$r \ll e^{r/2-1/(2r)} \tag{24}$$

$$\ln(r) \ll r/2 - 1/(2r) \tag{25}$$

$$4\ln(r) \ll 2r - 2/r \tag{26}$$

Hence, for all $\varepsilon > 0$

$$4\ln(r) < \varepsilon(2r - 2/r) \tag{27}$$

$$\frac{4\ln(r)}{2r - 2/r} < \varepsilon \tag{28}$$

$$\left|\left|\frac{4\ln(r)}{2r - 2/r}\right|\right| < \varepsilon \tag{29}$$

Since Equation 19 is true, $\mathbb{E}[f_r] = 0$. Note, if we simply took the average of f from $(-\infty, \infty)$, using the improper integral, the expected value:

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4 - x_3) + (x_2 - x_1)} \left(\int_{x_1}^{x_2} \frac{1}{x}\,dx + \int_{x_3}^{x_4} \frac{1}{x}\,dx\right) = \tag{30}$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4 - x_3) + (x_2 - x_1)} \left(\ln(||x||) + C\Big|_{x_1}^{x_2} + \ln(||x||) + C\Big|_{x_3}^{x_4}\right) = \tag{31}$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4 - x_3) + (x_2 - x_1)} \left(\ln(||x_2||) - \ln(||x_1||) + \ln(||x_4||) - \ln(||x_3||)\right) \tag{32}$$

is $+\infty$ (when $x_2 = 1/x_1$, $x_3 = 1/x_4$, and $x_1 = \exp\left(x_4^2\right)$) or $-\infty$ (when $x_2 = 1/x_1$, $x_3 = 1/x_4$, and $x_4 = -\exp\left(x_1^2\right)$), making $\mathbb{E}[f]$ undefined. (However, using Equation 12-19, we get the $\mathbb{E}[f_r] = 0$ instead of an undefined value.)

5.3. **Defining the "Measure".**

5.3.1. *Preliminaries.* We define the "**measure**" of $\{G_r^\star : r \in \mathcal{A}(A)\}$, in Section 5.3.3, which is the family of the graph of each $f_r^\star$ (Section 3.1.C). To understand this "measure", continue reading.

1. For every $r \in \mathcal{A}(A)$ (Section 2.3.1), "over-cover" $G_r^\star$ with minimal, pairwise disjoint sets of equal $\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}$ measure. (We denote the equal measures ε, where the former sentence is defined $\mathbf{C}(\varepsilon, G_r^\star, \omega)$: i.e., $\omega \in \Omega_{\varepsilon,r}$ enumerates all collections of these sets covering $G_r^\star$. In case this step is unclear, see Section 8.1. In addition, when there exists $r \in \mathcal{A}(A)$ such that $\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}(G_r^\star) = 0$, replace the Hausdorff measure $\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33].)
2. For every ε, r and ω, take a sample point from each set in $\mathbf{C}(\varepsilon, G_r^\star, \omega)$. The set of these points is "the sample" which we define $\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)$: i.e., $\psi \in \Psi_{\varepsilon,r,\omega}$ enumerates all possible samples of $\mathbf{C}(\varepsilon, G_r^\star, \omega)$. (If this is unclear, see Section 8.2.)
3. For every ε, r, ω and ψ,
   (a) Take a "pathway" of line segments:
      (i) start with a line segment from sample point $\mathbf{x}_0 \in \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)$ to the sample point $\mathbf{x}_1 \in \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi) \setminus \{\mathbf{x}_0\}$ with the smallest $(n+1)$-dimensional Euclidean distance to $\mathbf{x}_0$ (i.e., when more than one sample point has the smallest $(n+1)$-dimensional Euclidean distance to $\mathbf{x}_0$, take either of those points).
      (ii) Take a line segment from the sample point $\mathbf{x}_1 \in \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)$ to the sample point $\mathbf{x}_2 \in \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi) \setminus \{\mathbf{x}_0, \mathbf{x}_1\}$ with the smallest Euclidean distance to $\mathbf{x}_1$ (i.e., when more than one sample point has the smallest Euclidean distance to $\mathbf{x}_1$, take either of those points).
      (iii) Take a line segment from the sample point $\mathbf{x}_2 \in \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)$ to the sample point $\mathbf{x}_3 \in \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi) \setminus \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ with the smallest Euclidean distance to $\mathbf{x}_2$ (i.e., when more than one sample point has the smallest Euclidean distance to $\mathbf{x}_2$, take either of those points).
      (iv) Repeat this process until the pathway intersects with every sample point once. (In case this is unclear, see Section 8.3.1.)
   (b) Take the set of the length of all segments in (a), except for lengths that are outliers (i.e., for any constant $C > 0$, the outliers are more than C times the interquartile range of the length of all line segments as $r \to \infty$ or $\varepsilon \to 0$). Define this $\mathcal{L}(\mathbf{x}_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi))$. (If this is unclear, see Section 8.3.2.)
   (c) Multiply remaining lengths in the pathway by a constant so they add up to one (i.e., a probability distribution). This will be denoted $\mathbb{P}(\mathcal{L}(\mathbf{x}_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$. (In case this is unclear, see Section 8.3.3)

(d) Take the shannon entropy [11, p.61-95] of step (c). We define this:

$$\mathrm{E}(\mathbb{P}(\mathcal{L}(\mathbf{x}_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))) = \sum_{x \in \mathbb{P}(\mathcal{L}(\mathbf{x}_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))} -x \log_2 x$$

which will be *shortened* to $\mathrm{E}(\mathcal{L}(\mathbf{x}_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$. (If this is unclear, see Section 8.3.4.)

(e) Maximize the entropy w.r.t. all "pathways". This we will denote:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi))) = \sup_{\mathbf{x}_0 \in \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)} \mathrm{E}(\mathcal{L}(\mathbf{x}_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$$

(In case this is unclear, see Section 8.3.5.)

(4) Therefore, the **maximum entropy**, using (1) and (2) is:

$$\mathrm{E}_{\max}(\varepsilon, r) = \sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$$

5.3.2. *Generalized Definition of Limits.* Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f and Borel.

Note, $\lim_{x\to 0} f(x)$, $\limsup_{x\to 0} f(x)$, and $\liminf_{x\to 0} f(x)$ exist, when 0 is a limit of point of A. If 0 is not a limit point of A (e.g., $A = \mathbb{N}$), we change these definitions to include any A. The general definition of $\lim_{x\to 0} f(x)$ or $\overline{\lim}_{x\to 0} f(x)$ satisfies:

$$\forall(\epsilon_1 > \inf_{\epsilon \in A \cap \mathbb{R}^+}(\epsilon))\exists(\delta > 0)\forall(x \in A)\left(0 < |x| < \delta \Rightarrow \left|f(x) - \overline{\lim_{x\to 0}} f(x)\right| < \epsilon_1\right),$$

the general definition of $\limsup_{x\to 0} f(x)$ or $\overline{\limsup}_{x\to 0} f(x)$ satisfies:

$$\forall(\epsilon_2 > \inf_{\epsilon \in A \cap \mathbb{R}^+}(\epsilon))\exists(\delta > 0)\forall(t \in A \cap \mathbb{R}^+)\left(0 < |t| < \delta \Rightarrow \left|\sup\{f(x) : x \in A \cap (c-t, c+t)\} - \overline{\limsup_{x\to 0}} f(x)\right| < \epsilon_2\right),$$

and the general definition of $\liminf_{x\to 0} f(x)$ or $\overline{\liminf}_{x\to 0} f(x)$ satisfies:

$$\forall(\epsilon_3 > \inf_{\epsilon \in A \cap \mathbb{R}^+}(\epsilon))\exists(\delta > 0)\forall(t \in A \cap \mathbb{R}^+)\left(0 < |t| < \delta \Rightarrow \left|\inf\{f(x) : x \in A \cap (c-t, c+t)\} - \overline{\liminf_{x\to 0}} f(x)\right| < \epsilon_3\right)$$

5.3.3. *What Are We Measuring?* We define $\{G_r^\star : r \in \mathcal{A}(A)\}$ and $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$, which respectively are families of the graph for each of the bounded functions $f_r^\star$ and $f_v^{\star\star}$ (Section 3.1.C). Hence, for **constant** ε and *cardinality* $|\cdot|$

(a) Using (2) and (3e) of Section 5.3.1, suppose:

$$\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)|} =$$
$$\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon, v},\ \psi' \in \Psi_{\varepsilon, v, \omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \geq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))\right\}$$

then (using $\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)|}$) we get:

$$\overline{\alpha}(\varepsilon, r, \omega, \psi) = \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)|} / |\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi))|$$

(b) Also, using (2) and (3e) of Section 5.3.1, suppose:

$$\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)|} =$$
$$\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon, v},\ \psi' \in \Psi_{\varepsilon, v, \omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \leq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))\right\}$$

then (using $\underline{|\mathcal{S}(\mathbf{C}(\epsilon, G_r^\star, \omega), \psi)|}$) we also get

$$\underline{\alpha}(\varepsilon, r, \omega, \psi) = \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)|} / |\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi))|$$

(1) If using $\overline{\alpha}(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}(\epsilon, r, \omega, \psi)$ we have:[2]

$$1 < \overline{\limsup_{\varepsilon\to 0}} \limsup_{r\to\infty} \sup_{\omega \in \Omega_{\varepsilon, r}} \sup_{\psi \in \Psi_{\varepsilon, r, \omega}} \overline{\alpha}(\varepsilon, r, \omega, \psi), \overline{\liminf_{\varepsilon\to 0}} \liminf_{r\to\infty} \inf_{\omega \in \Omega_{\varepsilon, r}} \inf_{\psi \in \Psi_{\varepsilon, r, \omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty$$

then the "measure" of $\{G_r^\star : r \in \mathcal{A}(A)\}$ increases at a rate **superlinear** to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$.

---

[2] For the definitions of the notations $\overline{\limsup_{\varepsilon\to 0}}$ and $\overline{\liminf_{\varepsilon\to 0}}$, see Section 5.3.2

(2) If using equations $\overline{\alpha}\,(\varepsilon, v, \omega, \psi)$ and $\underline{\alpha}\,(\varepsilon, v, \omega, \psi)$ (where, using $\overline{\alpha}\,(\varepsilon, r, \omega, \psi)$ and $\underline{\alpha}\,(\varepsilon, r, \omega, \psi)$, we swap "$r \in \mathbb{N}$" with $v \in \mathbb{N}$ and $G_r^\star$ with $G_v^{\star\star}$) we get:[2]

$$1 < \mathop{\overline{\limsup}}\limits_{\varepsilon\to 0} \limsup_{v\to\infty} \sup_{\omega\in\Omega_{\varepsilon,v}} \sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \overline{\alpha}\,(\varepsilon, v, \omega, \psi)\,, \mathop{\overline{\liminf}}\limits_{\varepsilon\to 0} \liminf_{v\to\infty} \inf_{\omega\in\Omega_{\varepsilon,v}} \inf_{\psi\in\Psi_{\varepsilon,v,\omega}} \underline{\alpha}\,(\varepsilon, v, \omega, \psi) < +\infty$$

then the "measure" of $\{G_r^\star : r \in \mathcal{A}(A)\}$ increases at a rate **sublinear** to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$.

(3) If using equations $\overline{\alpha}\,(\varepsilon, r, \omega, \psi)$, $\underline{\alpha}\,(\varepsilon, r, \omega, \psi)$, $\overline{\alpha}\,(\varepsilon, v, \omega, \psi)$, and $\underline{\alpha}\,(\varepsilon, v, \omega, \psi)$, we **both** have:[2]

   (a) $\mathop{\overline{\limsup}}\limits_{\varepsilon\to 0} \limsup\limits_{r\to\infty} \sup\limits_{\omega\in\Omega_{\varepsilon,r}} \sup\limits_{\psi\in\Psi_{\varepsilon,r,\omega}} \overline{\alpha}\,(\varepsilon, r, \omega, \psi)$ or $\mathop{\overline{\liminf}}\limits_{\varepsilon\to 0} \liminf\limits_{r\to\infty} \inf\limits_{\omega\in\Omega_{\varepsilon,r}} \inf\limits_{\psi\in\Psi_{\varepsilon,r,\omega}} \underline{\alpha}\,(\varepsilon, r, \omega, \psi)$ are equal to zero, one or $+\infty$

   (b) $\mathop{\overline{\limsup}}\limits_{\varepsilon\to 0} \limsup\limits_{v\to\infty} \sup\limits_{\omega\in\Omega_{\varepsilon,v}} \sup\limits_{\psi\in\Psi_{\varepsilon,v,\omega}} \overline{\alpha}\,(\varepsilon, v, \omega, \psi)$ or $\mathop{\overline{\liminf}}\limits_{\varepsilon\to 0} \liminf\limits_{v\to\infty} \inf\limits_{\omega\in\Omega_{\varepsilon,v}} \inf\limits_{\psi\in\Psi_{\varepsilon,v,\omega}} \underline{\alpha}\,(\varepsilon, v, \omega, \psi)$ are equal to zero, one or $+\infty$

then the "measure" of $\{G_r^\star : r \in \mathcal{A}(A)\}$ increases at a rate **linear** to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$.

5.3.4. *Example of The "Measure" of $(G_r^\star)$ Increasing at Rate Super-linear to that of $(G_v^{\star\star})$.* Suppose, we have function $f : A \to \mathbb{R}$, where $A = \mathbb{Q} \cap [0,1]$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \end{cases} \tag{33}$$

such that (Section 2.3.1):

$$\{A_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\} = (\{c/r! : c \in \mathbb{Z}, 0 \leq c \leq r!\})_{r\in\mathbb{N}}$$

and

$$\{A_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\{c/d : c \in \mathbb{Z}, d \in \mathbb{N}, d \leq v, 0 \leq c \leq v\})_{v\in\mathbb{N}}$$

where for $f_r^\star : A_r^\star \to \mathbb{R}$,

$$f_r^\star(x) = f(x) \text{ for all } x \in A_r^\star \tag{34}$$

and $f_v^{\star\star} : A_v^{\star\star} \to \mathbb{R}$

$$f_v^{\star\star}(x) = f(x) \text{ for all } x \in A_v^{\star\star} \tag{35}$$

Hence, when $\{G_r^\star : r \in \mathcal{A}(A)\}$ (Section 2.3.1) is:

$$\{G_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\} = (\{(x, f_r^\star(x)) : x \in A_r^\star\})_{r\in\mathbb{N}} \tag{36}$$

and $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$ is:

$$\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\{(x, f_v^{\star\star}(x)) : x \in A_v^{\star\star}\})_{v\in\mathbb{N}} \tag{37}$$

Note, the following:

Since $\varepsilon > 0$ and $A = \mathbb{Q} \cap [0,1]$ is countably infinite, there exists a minimum ε which is 1. Thus, we do not need $\varepsilon \to 0$. Moreover, $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$ (Section 5.3.1 step 3e) is maximized by this procedure:

(1) For every $r \in \mathbb{N}$, group $(x, y) \in G_r^\star$ into $(x, f_r^\star(x))$, where x has an even denominator when simplified: i.e.,

$$S_{1,r} = \{(x, f_r^\star(x)) : x \in A_r^\star \cap \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1]\}$$

then group $(x, y) \in G_r^\star$ into $(x, f_r^\star(x))$, where x has an odd denominator when simplified: i.e.,

$$S_{2,r} = \{(x, f_r^\star(x)) : x \in A_r^\star \cap (\mathbb{Q} \setminus \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\}) \cap [0,1]\}$$

(2) Arrange the points in $S_{1,r}$ from least to greatest and take the 2-d Euclidean distance between each pair of consecutive points in $S_{1,r}$. Since the points lie on $y = 1$, take the absolute difference between each pair of consecutive x-coordinates of $S_{1,r}$ and call this $\mathcal{D}_{1,r}$. (Note, this is similar to Section 5.3.1 step 3a).

(3) Repeat step (2) for $S_{2,r}$, then call this $\mathcal{D}_{2,r}$. (Note, all points in $S_{2,r}$ lie on $y = 0$.)

(4) Remove any outliers from $\mathcal{D}_r = \mathcal{D}_{1,r} \cup \mathcal{D}_{2,r} \cup \{d((\frac{r!-1}{r!}, 1), (1, 0))\}$ (i.e., d is the 2-d Euclidean distance between the points $(\frac{r!-1}{r!}, 1)$ and $(1, 0)$). In this case, $\mathcal{D}_{2,r}$ and $d((\frac{r!-1}{r!}, 1), (1, 0))$ should be outliers (i.e., for any $C > 0$, the lengths in $\mathcal{D}_{2,r}$ and $\{d((\frac{r!-1}{r!}, 1), (1, 0))\}$ are more than C times the interquartile range of the lengths in $\mathcal{D}_r = \mathcal{D}_{1,r} \cup \mathcal{D}_{2,r} \cup \{d((\frac{r!-1}{r!}, 1), (1, 0))\}$ as $r \to \infty$) leaving us with $\mathcal{D}_{1,r}$.

(5) Multiply the remaining lengths in the pathway by a constant so that they add up to one. (See `P[r]` of Code 1 for an example)

(6) Take the entropy of the probability distribution in Step 5. (See `entropy[r]` of Code 1 for an example.)

We can illustrate this process with the following code:

CODE 1. Illustration of step (1)-(6)

```
(*We are using Mathematica*)

Clear["*Global'*"]
A[r_] := A[r] = Range[0, r!]/(r!)

(*Below is step 1*)
S1[r_] :=
 S1[r] = Sort[Select[A[r], Boole[IntegerQ[Denominator[#]/2]] == 1 &]]
S2[r_] :=
 S2[r] = Sort[Select[A[r], Boole[IntegerQ[Denominator[#]/2]] == 0 &]]

(*Below is step 2*)
Dist1[r_] := Dist1[r] = Differences[S1[r]]

(*Below is step 3*)
Dist2[r_] := Dist2[r] = Differences[S2[r]]

(*Below is step 4*)
NonOutliers[r_] :=
 NonOutliers[r] = Dist1[r] (*We exclude Dist2[r] since it's an outlier*)

(*Below is step 5*)
P[r_] := P[r] = NonOutliers[r]/Total[NonOutliers[r]]

(*Below is step 6*)
entropy[r_] := entropy[r] = Total[-P[r] Log[2, P[r]]]
```

Taking `Table[{r,entropy[r]},{r,3,8}]`, we get:

CODE 2. Output of `Table[{r,entropy[r]},{r,3,8}]`

```
Clear["*Global'*"]
{{{3,1}, {4,(2 Log[11])/(11 Log[2]) + (9 Log[22])/(11 Log[2])},
 {5,(14 Log[59])/(59 Log[2]) + (45 Log[118])/(59 Log[2])},
 {6,(44 Log[359])/(359 Log[2]) + (315 Log[718])/(359 Log[2])},
 {7,(314 Log[2519])/(2519 Log[2]) + (2205 Log[5038])/(2519 Log[2])},
 {8,(314 Log[20159])/(20159 Log[2]) + (19845 Log[40318])/(20159 Log[2])}}}
```

and note when:

1. $c(r) = (r!)/2 - 1$
2. $\{b(4) \mapsto 9,\ b(5) \mapsto 45,\ b(6) \mapsto 315,\ b(7) \mapsto 2205,\ b(8) \mapsto 19845\}$
3. $a(r) + b(r) = c(r)$

the output of Code 2 can be defined:

$$\frac{a(r)\log_2(c(r))}{c(r)} + \frac{b(r)\log(2c(r)))}{c(r)} = \frac{a(r)\log_2(c(r)) + b(r)\log(2c(r))}{c(r)} \tag{38}$$

Thus, since $a(r) = c(r) - b(r) = (r!)/2 - 1 - b(r)$:

$$\frac{a(r)\log_2(c(r)) + b(r)\log(2c(r))}{c(r)} = \tag{39}$$

$$\frac{(r!/2 - 1 - b(r))\log_2(c(r)) + b(r)\log_2(2c(r))}{c(r)} = \tag{40}$$

$$\frac{(r!/2)\log_2(c(r)) - \log_2(c(r)) - b(r)\log_2(r) + b(r)\log_2(c(r)) + b(r)\log_2(2)}{c(r)} = \tag{41}$$

$$\frac{(r!/2)\log_2(c(r)) - \log_2(c(r)) + b(r)}{c(r)} = \tag{42}$$

$$\frac{(r!/2 - 1)\log_2(c(r)) + b(r)}{c(r)} = \tag{43}$$

$$\frac{(r!/2-1)\log_2(r!/2-1)+b(r)}{r!/2-1} = \tag{44}$$

$$\log_2(r!/2-1)+\frac{b(r)}{r!/2-1} = \tag{45}$$

and $\lim_{r\to\infty} b(r)/c(r) = 1$ (we need help proving this):

$$\log_2(r!/2-1)+\frac{b(r)}{r!/2-1} \sim \log_2(r!/2-1)+1 \tag{46}$$

$$\log_2(r!/2-1)+\log_2(2) = \tag{47}$$

$$\log_2(2(r!/2-1)) \tag{48}$$

$$\log_2(r!-2) \sim \log_2(r!) \tag{49}$$

Hence, `entropy[r]` is the same as:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1,G_r^{\star},\omega),\psi))) \sim \log_2(r!) \tag{50}$$

Now, repeat Code 1 with (Section 2.3.1):

$$\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\{(x, f_v^{\star\star}(x)) : x \in A_v^{\star\star} := (\{c/d : c \in \mathbb{Z}, d \in \mathbb{N}, d \le v, 0 \le c \le v\})_{v\in\mathbb{N}}\})_{v\in\mathbb{N}}$$

Code 3. Illustration of step (1)-(6) on $\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\}$

```
(*We are using Mathematica*)

Clear[”*Global‘*”]
A[v_] := A[v] =
  DeleteDuplicates[Flatten[Table[Range[0, t]/t, {t, 1, v}]]]

(*Below is step 1*)
S1[v_] :=
 S1[v] = Sort[Select[A[v], Boole[IntegerQ[Denominator[#]/2]] == 1 &]]
S2[v_] :=
 S2[v] = Sort[Select[A[v], Boole[IntegerQ[Denominator[#]/2]] == 0 &]]

(*Below is step 2*)
Dist1[v_] := Dist1[v] = Differences[S1[v]]

(*Below is step 3*)
Dist2[v_] := Dist2[v] = Differences[S2[v]]

(*Below is step 4*)
NonOutliers[v_] :=
 NonOutliers[v] = Join[Dist1[v], Dist2[v]] (*There are no outliers*)

(*Below is step 5*)
P[v_] := P[v] = NonOutliers[v]/Total[NonOutliers[v]]

(*Below is step 6*)

entropy[v_] := entropy[v] = N[Total[-P[v] Log[2, P[v]]]]
```

Using this post [18], we assume an approximation of `Table[entropy[v],{v,3,Infinity}]` or $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1,G_v^{\star\star},\omega'),\psi')))$ is:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1,G_v^{\star\star},\omega'),\psi'))) \sim 2\log_2(v)+1-\log_2(3\pi) \tag{51}$$

Thus, using Section 5.3.3 (a) and Section 5.3.3 (1), take $|\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')| = \sum_{M=1}^{v}\phi(M) \approx \frac{3}{\pi^2}v^2$ (where ϕ is Euler's Totient function) to compute the following:

$$\begin{aligned}&\overline{|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)|} = \\ &\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')\right| : v\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,v},\ \psi'\in\Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')))\ge \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)))\right\} = \\ &\inf\left\{\frac{3}{\pi^2}v^2 : r\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,v},\ \psi'\in\Psi_{\varepsilon,v,\omega},\ 2\log_2(v)+1-\log_2(3\pi)\ge\log_2(r!)\right\} =\end{aligned} \tag{52}$$

where:

(1) For every $r \in \mathbb{N}$, we find a $v \in \mathbb{N}$, where $2\log_2(v) + 1 - \log_2(3\pi) \geq \log_2(r!)$, but the absolute value of $(2\log_2(v) + 1 - \log_2(3\pi)) - \log_2(r!)$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $v \in \mathbb{N}$ where:

$$2\log_2(v) + 1 - \log_2(3\pi) \geq \log_2(r!) \tag{53}$$

$$2^{2\log_2(v)} \geq \log_2(r!) - 1 + \log_2(3\pi) \tag{54}$$

$$\left(2^{\log_2(v)}\right)^2 \geq 2^{\log_2(r!)-1+\log_2(3\pi)} \tag{55}$$

$$v^2 \geq \left(2^{\log_2(r!)}2^{\log_2(3\pi)}\right)/2 \tag{56}$$

$$v \geq \sqrt{\frac{r!(3\pi)}{2}} \tag{57}$$

$$v = \left\lceil \sqrt{\frac{3\pi r!}{2}} \right\rceil \tag{58}$$

$$\frac{3}{\pi^2}v^2 = \frac{3}{\pi^2}\left(\left\lceil \sqrt{\frac{3\pi r!}{2}} \right\rceil\right)^2 \sim \overline{|\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi)|} \tag{59}$$

Finally, since $|\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi)| = r!$, we wish to prove[3]

$$1 < \overline{\limsup_{\varepsilon \to 0}} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}\left(\varepsilon, r, \omega, \psi\right) < +\infty$$

with Section 5.3.3 crit. 1:

$$\overline{\limsup_{\varepsilon \to 0}} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}\left(\varepsilon, r, \omega, \psi\right) = \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \frac{\overline{|\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi)|}}{|\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi))|} \tag{60}$$

$$= \lim_{r \to \infty} \frac{\frac{3}{\pi^2}\left(\left\lceil \sqrt{\frac{3\pi r!}{2}} \right\rceil\right)^2}{r!} \tag{61}$$

where, from Mathematica, we get the limit in Equation 61 is greater than one:

Code 4. Limit of Equation 61

```
N[Limit[((3/Pi^2) (Ceiling[Sqrt[(3 Pi r!)/2]])^2)/(r!), r -> Infinity]]

(*The output is 1.43239*)
```

Also, using Section 5.3.3 (b) and Section 5.3.3 (1), take $|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')| = \sum_{M=1}^{v} \phi(M) \approx \frac{3}{\pi^2}v^2$ (where ϕ is Euler's Totient function) computing the following:

$$\begin{aligned} &\underline{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)\right|} = \\ &\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \leq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))\right\} = \\ &\sup\left\{\frac{3}{\pi^2}v^2 : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ 2\log_2(v) + 1 - \log_2(3\pi) \leq \log_2(r!)\right\} = \end{aligned} \tag{62}$$

where:

(1) For every $r \in \mathbb{N}$, we find a $v \in \mathbb{N}$, where $2\log_2(r) + 1 - \log_2(3\pi) \leq \log_2(r!)$, but the absolute value of $\log_2(r!) - (2\log_2(v) + 1 - \log_2(3\pi))$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $v \in \mathbb{N}$

---

[3] For a definition of the notation $\overline{\limsup_{\varepsilon \to 0}}$, see Section 5.3.2

where:

$$2\log_2(v) + 1 - \log_2(3\pi) \le \log_2(r!) \tag{63}$$

$$2\log_2(v) \le \log_2(r!) - 1 + \log_2(3\pi) \tag{64}$$

$$\left(2^{\log_2(v)}\right)^2 \le 2^{\log_2(r!)-1+\log_2(3\pi)} \tag{65}$$

$$(v)^2 \le \left(2^{\log_2(r!)}2^{\log_2(3\pi)}\right)/2 \tag{66}$$

$$v \le \sqrt{\frac{r!(3\pi)}{2}} \tag{67}$$

$$v = \left\lfloor\sqrt{\frac{3\pi r!}{2}}\right\rfloor \tag{68}$$

$$\frac{3}{\pi^2}v^2 = \frac{3}{\pi^2}\left(\left\lfloor\sqrt{\frac{3\pi r!}{2}}\right\rfloor\right)^2 \sim \underline{|\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi)|} \tag{69}$$

Finally, since $|\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi)| = r!$, we wish to prove[4]

$$1 < \overline{\liminf_{\varepsilon\to 0}} \liminf_{r\to\infty} \inf_{\omega\in\Omega_{\varepsilon,r}} \inf_{\psi\in\Psi_{\varepsilon,r,\omega}} \underline{\alpha}\left(\varepsilon, r, \omega, \psi\right) < +\infty$$

with Section 5.3.3 crit. 1:

$$\overline{\liminf_{\varepsilon\to 0}} \liminf_{r\to\infty} \inf_{\omega\in\Omega_{\varepsilon,r}} \inf_{\psi\in\Psi_{\varepsilon,r,\omega}} \underline{\alpha}\left(\varepsilon, r, \omega, \psi\right) = \liminf_{r\to\infty} \inf_{\omega\in\Omega_{\varepsilon,r}} \inf_{\psi\in\Psi_{\varepsilon,r,\omega}} \frac{\underline{|\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi)|}}{|\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi))|} \tag{70}$$

$$= \lim_{r\to\infty} \frac{\frac{3}{\pi^2}\left(\left\lfloor\sqrt{\frac{3\pi r!}{2}}\right\rfloor\right)^2}{r!} \tag{71}$$

where, from Mathematica, we get the limit is greater than one:

Code 5. Limit of Equation 71

```
Clear["*Global'*"]
N[Limit[((3/Pi^2) (Floor[Sqrt[(3 Pi r!)/2]])^2)/(r!), r -> Infinity]]

 (*Output is 1.43239*)
```

Hence, since the limits in Equation 61 and Equation 71 are greater than one and less than $+\infty$: i.e., [5]

$$1 < \overline{\liminf_{\varepsilon\to 0}} \liminf_{r\to\infty} \inf_{\omega\in\Omega_{\varepsilon,r}} \inf_{\psi\in\Psi_{\varepsilon,r,\omega}} \underline{\alpha}\left(\varepsilon, r, \omega, \psi\right) = \overline{\limsup_{\varepsilon\to 0}} \limsup_{r\to\infty} \sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \overline{\alpha}\left(\varepsilon, r, \omega, \psi\right) < +\infty \tag{72}$$

then the "measure" of $\{G_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\}$ increases at a rate **superlinear** to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\}$ (i.e., 5.3.3 crit. 1).

5.3.5. *Example of The "Measure" from $\{G_r^\star : r \in \mathcal{A}(A)\}$ Increasing at a Rate Sub-Linear to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$.*

Using our previous example, we can use the following theorem:

Theorem 7. *If the "measure" of $\{G_r^\star : r \in \mathcal{A}(A)\}$ increases at a rate superlinear to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$, then the "measure" of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$ increases at a rate **sublinear** to that of $\{G_r^\star : r \in \mathcal{A}(A)\}$*

Hence, in our definition of super-linear (Section 5.3.3 crit. 1), swap $G_r^\star$ for $G_v^{\star\star}$ *and* $r \in \mathbb{N}$ for $v \in \mathbb{N}$ regarding $\overline{\alpha}\left(\epsilon, r, \omega, \psi\right)$ and $\underline{\alpha}\left(\epsilon, r, \omega, \psi\right)$ (i.e., $\overline{\alpha}\left(\epsilon, v, \omega, \psi\right)$ and $\underline{\alpha}\left(\epsilon, v, \omega, \psi\right)$) and notice Theorem 7 is true when:[5]

$$1 < \overline{\limsup_{\varepsilon\to 0}} \limsup_{v\to\infty} \sup_{\omega\in\Omega_{\varepsilon,v}} \sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \overline{\alpha}\left(\varepsilon, v, \omega, \psi\right), \overline{\liminf_{\varepsilon\to 0}} \liminf_{v\to\infty} \inf_{\omega\in\Omega_{\varepsilon,v}} \inf_{\psi\in\Psi_{\varepsilon,v,\omega}} \underline{\alpha}\left(\varepsilon, v, \omega, \psi\right) < +\infty$$

---

[4] For a definition of the notation $\overline{\liminf_{\varepsilon\to 0}}$, see Section 5.3.2

[5] For the definitions of the notations $\overline{\limsup_{\varepsilon\to 0}}$ and $\overline{\liminf_{\varepsilon\to 0}}$, see Section 5.3.2

5.3.6. *Example of The "Measure" from $\{G_r^{\star} : r \in \mathcal{A}(A)\}$ Increasing at a Rate Linear to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$.* Suppose, we have function $f : A \to \mathbb{R}$, where $A = \mathbb{Q} \cap [0,1]$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \end{cases} \tag{73}$$

such that (Section 2.3.1):

$$\{A_r^{\star} : r \in \mathcal{A}(A) := \mathbb{N}\} = (\{c/r! : c \in \mathbb{Z}, 0 \leq c \leq r!\})_{r \in \mathbb{N}}$$

and

$$\{A_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\left\{c/(v!)^2 : c \in \mathbb{N}, 1 \leq c \leq (v!)^2\right\})_{v \in \mathbb{N}}$$

where for $f_r^{\star} : A_r^{\star} \to \mathbb{R}$,

$$f_r^{\star}(x) = f(x) \text{ for all } x \in A_r^{\star} \tag{74}$$

and $f_v^{\star\star} : A_v^{\star\star} \to \mathbb{R}$

$$f_v^{\star\star}(x) = f(x) \text{ for all } x \in A_v^{\star\star} \tag{75}$$

Hence, when $\{G_r^{\star} : r \in \mathcal{A}(A)\}$ (Section 2.3.1) is:

$$\{G_r^{\star} : r \in \mathcal{A}(A) := \mathbb{N}\} = (\{(x, f_r^{\star}(x)) : x \in A_r^{\star}\})_{r \in \mathbb{N}} \tag{76}$$

and $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$ is:

$$\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\{(x, f_v^{\star\star}(x)) : x \in A_v^{\star\star}\})_{v \in \mathbb{N}} \tag{77}$$

We know, using Equation 50:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi))) \sim \log_2(r! - 2) \sim \log_2(r!) \tag{78}$$

Also, using Section 5.3.4 steps 1-6 on $\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\}$:

Code 6. Illustration of step (1)-(6) on $\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\}$

```
(*We are using Mathematica*)

Clear["*Global`*"]
A[v_] := A[v] = Range[0, 7 (v!)]/(7 (v!))

(*Below is step 1*)
S1[v_] :=
 S1[v] = Sort[Select[A[v], Boole[IntegerQ[Denominator[#]/2]] == 1 &]]
S2[v_] :=
 S2[v] = Sort[Select[A[v], Boole[IntegerQ[Denominator[#]/2]] == 0 &]]

(*Below is step 2*)
Dist1[v_] := Dist1[v] = Differences[S1[v]]

(*Below is step 3*)
Dist2[v_] := Dist2[v] = Differences[S2[v]]

(*Below is step 4*)
NonOutliers[v_] :=
 NonOutliers[v] = Dist1[v] (*Dist2[v] is an outlier*)

(*Below is step 5*)
P[v_] := P[v] = NonOutliers[v]/Total[NonOutliers[v]]

(*Below is step 6*)

entropy[v_] := entropy[v] = N[Total[-P[v] Log[2, P[v]]]]

T = Table[{v, entropy[v]}, {v, 3, 6}]
```

where the output is

Code 7. Output of Code 6

```
{{3,(8 Log[17])/(17 Log[2]) + (9 Log[34])/(17 Log[2])},
     {4,(8 Log[287])/(287 Log[2]) + (279 Log[574])/(287 Log[2])},
     {5,(224 Log[7199])/(7199 Log[2]) + (6975 Log[14398])/(7199 Log[2])},
     {6,(2024 Log[259199])/(259199 Log[2]) + (257175 Log[518398])/(259199 Log[2])}}
```

Note, when:

1. $c(v) = (v!)^2/2 - 1$
2. $\{b(4) \mapsto 9,\, b(5) \mapsto 279,\, b(6) \mapsto 6975,\, b(7) \mapsto 257175,\, b(8) \mapsto 19845\}$
3. $a(v) + b(v) = c(v)$

Code 7 can be defined:

$$\frac{a(v)\log_2(c(v))}{c(v)} + \frac{b(v)\log(2c(v))}{c(v)} = \frac{a(v)\log_2(c(v)) + b(v)\log(2c(v))}{c(v)} \tag{79}$$

Thus, since $a(v) = c(v) - b(v) = (v!)^2/2 - 1 - b(v)$:

$$\frac{a(v)\log_2(c(v)) + b(v)\log(2c(v))}{c(v)} = \tag{80}$$

$$\frac{((v!)^2/2 - 1 - b(v))\log_2(c(v)) + b(v)\log_2(2c(v))}{c(v)} = \tag{81}$$

$$\frac{((v!)^2/2)\log_2(c(v)) - \log_2(c(v)) - b(v)\log_2(v) + b(v)\log_2(c(v)) + b(v)\log_2(2)}{c(v)} = \tag{82}$$

$$\frac{((v!)^2/2)\log_2(c(v)) - \log_2(c(v)) + b(v)}{c(v)} = \tag{83}$$

$$\frac{((v!)^2/2 - 1)\log_2(c(v)) + b(v)}{c(v)} = \tag{84}$$

$$\frac{((v!)^2/2 - 1)\log_2((v!)^2/2 - 1) + b(v)}{(v!)^2/2 - 1} = \tag{85}$$

$$\log_2((v!)^2/2 - 1) + \frac{b(v)}{(v!)^2/2 - 1} = \tag{86}$$

and $\lim_{v\to\infty} b(v)/c(v) = 1$ (this is proven in [19]):

$$\log_2((v!)^2/2 - 1) + \frac{b(v)}{(v!)^2/2 - 1} \sim \log_2((v!)^2/2 - 1) + 1 \tag{87}$$

$$\log_2((v!)^2/2 - 1) + \log_2(2) = \tag{88}$$

$$\log_2((v!)^2 - 2)) \sim \tag{89}$$

$$\log_2((v!)^2) = \tag{90}$$

$$2\log_2(v!) \tag{91}$$

Then, `entropy[r]` is the same as:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_v^{\star\star}, \omega), \psi))) \sim \tag{92}$$

$$2\log_2(v!) \tag{93}$$

Therefore, using Section 5.3.3 (b) and Section 5.3.3 (3a), take $|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')| = (v!)^2$ to compute the following:

$$\underline{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)\right|} = \tag{94}$$

$$\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \leq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))\right\} =$$

$$\sup\left\{(v!)^2 : r \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ 2\log_2(v!) \leq \log_2(r!)\right\} =$$

where:

(1) For every $r \in \mathbb{N}$, we find a $v \in \mathbb{N}$, where $2\log_2(v!) \leq \log_2(r!)$, but the absolute value of $\log_2(r!) - 2\log_2(v!)$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $v \in \mathbb{N}$ where:

$$2\log_2(v!) \leq \log_2(r!) \tag{95}$$

$$2^{2\log_2(v!)} \leq 2^{\log_2(r!)} \tag{96}$$

$$(2^{\log_2(v!)})^2 \leq r! \tag{97}$$

$$(v!)^2 \leq r! \tag{98}$$

$$(v!)^2 = \lfloor r! \rfloor \tag{99}$$

To solve for v, we try the following code:

Code 8. Code for v in Equation 99

```
(*We are using Mathematica*)

Clear["Global`*"]

T1 = Table[
   {sol[r_] := sol[r] = Reduce[v > 0 && ((v!)^2) <= r!, v, Integers],
    vsolve = Max[v /. Solve[sol[r], {v}, Integers]],
    (* Largest v that solves inequality (v!)^2<=r for every r *)
    , N[(vsolve!)^2/(r!)]}, {r, 3, 40}];

Tablevsolve =
 Table[{T1[[r - 3 + 1, 2]], r}, {r, 3,
   40}] (*Takes largest v-values for every r in r!*)

loweralphr =
 Table[{r, T1[[r - 3 + 1, 4]]}, {r, 3,
   40}] (* Takes largest largest v-values and corresponding r value*)

ListPlot[loweralphr] (*Graph points of loweralphr. Notice, the graph has
a lower bound of zero.*)
```

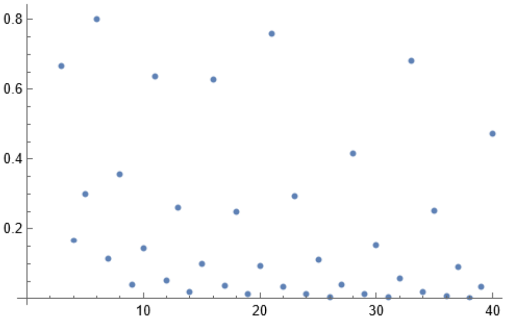
Note, the output is:

Code 9. Output for Code 8

```
Clear["Global`*"]
(* Output of Tablevsolve*)
{{2, 3}, {2, 4}, {3, 5}, {4, 6}, {4, 7}, {5, 8}, {5, 9}, {6, 10}, {7, 11}, {7, 12},
 {8, 13}, {8, 14}, {9, 15}, {10, 16}, {10, 17}, {11, 18}, {11, 19}, {12, 20}, {13, 21},
 {13, 22}, {14, 23}, {14, 24}, {15, 25}, {15, 26}, {16, 27}, {17, 28}, {17, 29},
 {18, 30}, {18, 31}, {19, 32}, {20, 33}, {20, 34}, {21, 35}, {21, 36}, {22, 37}, {22, 38},
 {23, 39}, {24, 40}}

(*Output of loweralphr*)
{{3, 0.666667}, {4, 0.166667}, {5, 0.3}, {6, 0.8}, {7, 0.114286}, {8, 0.357143}, {9, 0.0396825},
 {10, 0.142857}, {11, 0.636364}, {12, 0.0530303}, {13, 0.261072}, {14, 0.018648}, {15, 0.100699},
 {16, 0.629371}, {17, 0.0370218}, {18, 0.248869}, {19, 0.0130984}, {20, 0.0943082}, {21, 0.758956},
 {22, 0.034498}, {23, 0.293983}, {24, 0.0122493}, {25, 0.110244}, {26, 0.00424014}, {27,0.0402028},
 {28, 0.41495}, {29, 0.0143086}, {30, 0.154533}, {31, 0.00498494}, {32, 0.0562364}, {33, 0.681653},
 {34, 0.0200486}, {35, 0.252613}, {36, 0.00701702}, {37, 0.0917902}, {38, 0.00241553},
 {39, 0.0327645}, {40, 0.471809}}
```

FIGURE 1. Plot of `loweralphr`



Finally, since the lower bound of `loweralphr` is zero, we have shown:[6]

$$\overline{\liminf_{\varepsilon\to 0}}\,\liminf_{r\to\infty}\,\inf_{\omega\in\Omega_{\varepsilon,r}}\,\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\,\underline{\alpha}\,(\varepsilon,r,\omega,\psi)=0 \tag{100}$$

Next, using Section 5.3.3 (b) and Section 5.3.3 (3b), take $|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega'),\psi')|=r!$ and swap $r\in\mathbb{N}$ and $\{G_r^{\star}:r\in\mathcal{A}(A):=\mathbb{N}\}$ with $v\in\mathbb{N}$ and $\{G_v^{\star\star}:v\in\mathcal{A}(A):=\mathbb{N}\}$, to compute the following:

$$\begin{aligned}&\underline{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega),\psi)\right|}= \qquad\qquad (101)\\&\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega'),\psi')\right|:r\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,r},\ \psi'\in\Psi_{\varepsilon,r,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega'),\psi')))\geq\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega),\psi)))\right\}=\\&\inf\left\{r!:r\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,r},\ \psi'\in\Psi_{\varepsilon,r,\omega},\ \log_2(r!)\geq 2\log_2(v!)\right\}=\end{aligned}$$

where:

(1) For every $v\in\mathbb{N}$, we find a $r\in\mathbb{N}$, where $\log_2(r!)\leq 2\log_2(v!)$, but the absolute value of $2\log_2(v!)-\log_2(r!)$ is minimized. In other words, for every $v\in\mathbb{N}$, we want $r\in\mathbb{N}$ where:

$$\log_2(r!)\leq 2\log_2(v!) \tag{102}$$

$$2^{\log_2(r!)}\leq 2^{2\log_2(v!)} \tag{103}$$

$$r!\leq(2^{\log_2(v!)})^2 \tag{104}$$

$$r!\leq(v!)^2 \tag{105}$$

$$r!=(v!)^2 \tag{106}$$

To solve r, we try the following code:

CODE 10. Code for r in Equation 106

```
(*We are using Mathematica*)

Clear[”Global‘*”]
T2 = Table[
   {sol[v_] := sol[v] = Reduce[v > 0 && r! <= (v!)^2, r, Integers],
    rsolve = Max[r /. Solve[sol[v], {r}, Integers]],
    (* Largest r that solves inequality (r!)<=(v!)^2 for every v *)
    , N[(rsolve!)/((v!)^2)]}, {v, 3, 40}];

Tablersolve =
 Table[{T2[[v - 3 + 1, 2]], v}, {v, 3,
   40}] (*Takes largest r-values for every v in (v!)^2*)

loweralphv =
 Table[{v, T2[[v - 3 + 1, 4]]}, {v, 3,
   40}] (* Takes largest largest r values and corresponding v value*)

ListPlot[loweralphv](*Graph points of loweralphv. Notice, the graph
has a lower bound of zero*)
```

---

[6] For a definition of the notation $\overline{\liminf_{\varepsilon\to 0}}$, see Section 5.3.2

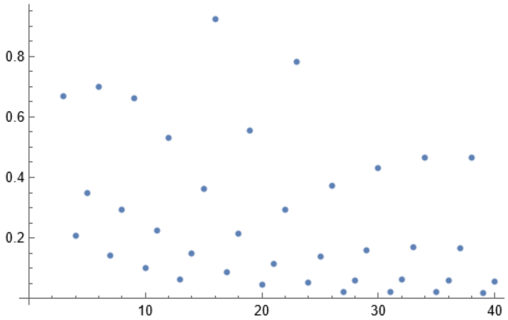
Note, the output is:

Code 11. Output for Code 10

```
Clear["Global`*"]
(* Output of Tablersolve*)
{{4, 3}, {5, 4}, {7, 5}, {9, 6}, {10, 7}, {12, 8}, {14, 9}, {15, 10}, {17, 11}, {19, 12},
 {20, 13}, {22, 14}, {24, 15}, {26, 16}, {27, 17}, {29, 18}, {31, 19}, {32, 20}, {34, 21},
 {36, 22}, {38, 23}, {39, 24}, {41, 25}, {43, 26}, {44, 27}, {46, 28}, {48, 29}, {50, 30},
 {51, 31}, {53, 32}, {55, 33}, {57, 34}, {58, 35}, {60, 36}, {62, 37}, {64, 38}, {65, 39},
 {67, 40}}

(*Output of loweralphv*)
{{3, 0.666667}, {4, 0.166667}, {5, 0.3}, {6, 0.8}, {7, 0.114286}, {8, 0.357143}, {9, 0.0396825},
 {10, 0.142857}, {11, 0.636364}, {12, 0.0530303}, {13, 0.261072}, {14, 0.018648}, {15, 0.100699},
 {16, 0.629371}, {17, 0.0370218}, {18, 0.248869}, {19, 0.0130984}, {20, 0.0943082}, {21, 0.758956},
 {22, 0.034498}, {23, 0.293983}, {24, 0.0122493}, {25, 0.110244}, {26, 0.00424014}, {27,0.0402028},
 {28, 0.41495}, {29, 0.0143086}, {30, 0.154533}, {31, 0.00498494}, {32, 0.0562364}, {33, 0.681653},
 {34, 0.0200486}, {35, 0.252613}, {36, 0.00701702}, {37, 0.0917902}, {38, 0.00241553},
 {39, 0.0327645}, {40, 0.471809}}
```

Figure 2. Plot of `loweralphv`



since the lower bound of `loweralphv` is zero, we have shown:[7]

$$\overline{\liminf_{\varepsilon\to 0}}\,\liminf_{v\to\infty}\,\inf_{\omega\in\Omega_{\varepsilon,v}}\,\inf_{\psi\in\Psi_{\varepsilon,v,\omega}}\,\underline{\alpha}\left(\varepsilon,v,\omega,\psi\right)=0 \tag{107}$$

Hence, using Equation 100 and 107, since **both**:[8]

(1) $\overline{\limsup_{\varepsilon\to 0}}\,\limsup_{r\to\infty}\,\sup_{\omega\in\Omega_{\varepsilon,r}}\,\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\,\overline{\alpha}\left(\varepsilon,r,\omega,\psi\right)$ or $\overline{\liminf_{\varepsilon\to 0}}\,\liminf_{r\to\infty}\,\inf_{\omega\in\Omega_{\varepsilon,r}}\,\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\,\underline{\alpha}\left(\varepsilon,r,\omega,\psi\right)$ are equal to zero, one or $+\infty$

(2) $\overline{\limsup_{\varepsilon\to 0}}\,\limsup_{v\to\infty}\,\sup_{\omega\in\Omega_{\varepsilon,v}}\,\sup_{\psi\in\Psi_{\varepsilon,v,\omega}}\,\overline{\alpha}\left(\varepsilon,v,\omega,\psi\right)$ or $\overline{\liminf_{\varepsilon\to 0}}\,\liminf_{v\to\infty}\,\inf_{\omega\in\Omega_{\varepsilon,v}}\,\inf_{\psi\in\Psi_{\varepsilon,v,\omega}}\,\underline{\alpha}\left(\varepsilon,v,\omega,\psi\right)$ are equal to zero, one or $+\infty$

the "measure" of $\{G_r^{\star}:r\in\mathcal{A}(A)\}$ increases at a rate **linear** to that of $\{G_v^{\star\star}:v\in\mathcal{A}(A)\}$.

5.4. Defining The Actual Rate of Expansion of a family of Bounded Sets.

5.4.1. *Definition of Actual Rate of Expansion of a family of Bounded Sets.* Suppose:

1. $\{G_r^{\star}:r\in\mathcal{A}(A)\}$ is a family of the graph of each $f_r^{\star}$ (Section 3.1.C)
2. C is a reference point in $\mathbb{R}^{n+1}$
3. $Q,R\in\mathbb{R}^{n+1}$
4. $Q=(q_1,\cdot\cdot\cdot,q_{n+1})$ and $R=(r_1,\cdot\cdot\cdot,r_{n+1})$, where:

$$Q-R=(q_1-r_1,\cdot\cdot\cdot,q_{n+1}-r_{n+1})$$

---

[7] For a definition of the notation $\overline{\liminf_{\varepsilon\to 0}}$, see Section 5.3.2

[8] For the definitions of the notations $\overline{\liminf_{\varepsilon\to 0}}$ and $\overline{\limsup_{\varepsilon\to 0}}$, see Section 5.3.2

(5) $||Q||_{n+1} = \sqrt{q_1^2 + \cdots + q_{n+1}^2}$ and $||R||_{n+1} = \sqrt{r_1^2 + \cdots + r_{n+1}^2}$
(6) $C - G_r^\star = \{C - y : y \in G_r^\star\}$
(7) $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension
(8) $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra

For any $r \in \mathbb{N}$, take the $(n+1)$-dimensional Euclidean distance between a reference point $C \in \mathbb{R}^{n+1}$ and each point in $G_r^\star$:

$$\mathcal{G}(C, G_r^\star) = \{||C - y||_{n+1} : y \in G_r^\star\}$$

then average $\mathcal{G}(C, G_r^\star)$:

$$\mathrm{Avg}(\mathcal{G}(C, G_r^\star)) = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(C - G_r^\star)}(C - G_r^\star)} \int_{C - G_r^\star} ||(x_1, \cdots, x_{n+1})||_{n+1} \, d\mathcal{H}^{\dim_{\mathrm{H}}(C - G_r^\star)}$$

where **the actual rate of expansion of** $\{G_r^\star : r \in \mathcal{A}(A)\}$ is:[9]

$$\mathcal{E}(C, G_r^\star) = \overline{\lim_{h \to 0}} \frac{\mathrm{Avg}(\mathcal{G}(C, G_{r+h}^\star)) - \mathrm{Avg}(\mathcal{G}(C, G_r^\star))}{h} \tag{108}$$

If $\mathcal{E}(C, G_r^\star)$ is undefined, replace the Hausdorff measure $\mathcal{H}^{\dim_{\mathrm{H}}(C - G_r^\star)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33]

5.4.2. *Example.* Suppose, we have $f : A \to \mathbb{R}$, where $A = \mathbb{R}$ and $f(x) = x$, such that $\{A_r^\star : r \in \mathcal{A}(A) := \mathbb{R}^+\} = \{[-r, r] : r \in \mathbb{R}^+\}$ and for $f_r^\star : A_r^\star \to \mathbb{R}$:

$$f_r^\star(x) = f(x) \text{ for all } x \in A_r^\star$$

Hence, when $\{G_r^\star : r \in \mathcal{A}(A)\}$ (Section 2.3.1) is:

$$\{G_r^\star : r \in \mathcal{A}(A) := \mathbb{R}^+\} = \left\{\{(x, x) : x \in [-r, r]\} : r \in \mathbb{R}^+\right\}$$

such that $C = (0, 0)$, note:

$$\mathrm{Avg}(\mathcal{G}(C, G_r^\star)) = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(C - G_r^\star)}(C - G_r^\star)} \int_{C - G_r^\star} ||(x_1, x_2)||_2 \, d\mathcal{H}^{\dim_{\mathrm{H}}(C - G_r^\star)} \tag{109}$$

$$= \frac{1}{\mathcal{H}^1((0,0) - G_r^\star)} \int_{-r}^{r} ||(x_1, x_1)||_2 \, d\mathcal{H}^1 = \text{ (since } x_2 = f(x_1) = x_1) \tag{110}$$

$$= \frac{1}{\mathrm{length}(G_r^\star)} \int_{-r}^{r} \sqrt{x_1^2 + x_1^2} \, dx_1 \tag{111}$$

$$= \frac{1}{\sqrt{(r - (-r))^2 - (r - (-r))^2}} \int_{-r}^{r} \sqrt{2x_1^2} \, dx_1 \tag{112}$$

$$= \frac{1}{\sqrt{(2r)^2 + (2r)^2}} \int_{-r}^{r} \sqrt{2x_1^2} \, dx_1 \tag{113}$$

$$= \frac{1}{2\sqrt{2}r} \int_{-r}^{r} \sqrt{2}|x_1| \, dx_1 \tag{114}$$

$$= \frac{1}{2\sqrt{2}r} \left(\frac{\sqrt{2}}{2} \mathrm{sign}(x_1)(x_1)^2 \Big|_{-r}^{r} \right) \tag{115}$$

$$= \frac{1}{2\sqrt{2}r} \left(\frac{\sqrt{2}}{2} \mathrm{sign}(r) r^2 - \frac{\sqrt{2}}{2} \mathrm{sign}(-r)(-r)^2 \right) \tag{116}$$

$$= \frac{1}{2\sqrt{2}r} \left(\frac{\sqrt{2}}{2} r^2 + \frac{\sqrt{2}}{2} r^2 \right) \tag{117}$$

$$= \frac{1}{2\sqrt{2}r} \left(\sqrt{2} r^2 \right) \tag{118}$$

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[9] For a definition of the notation $\overline{\lim_{\varepsilon \to 0}}$, see Section 5.3.2

$$=\frac{1}{2}r \tag{119}$$

$$\tag{120}$$

and the actual rate of expansion is:[10]

$$\mathcal{E}(C,G_r^{\star})=\overline{\lim_{h\to 0}}\ \frac{\operatorname{Avg}(\mathcal{G}(C,G_{r+h}^{\star}))-\operatorname{Avg}(\mathcal{G}(C,G_r^{\star}))}{h}= \tag{121}$$

$$\overline{\lim_{h\to 0}}\ \frac{\frac{1}{2}(r+h)-\frac{1}{2}r}{h} \tag{122}$$

Since $h\in\mathbb{R}$, hence Equation 122 is the same as the limit definition of the derivative of $\frac{1}{2}r$. Thus,

$$\mathcal{E}(C,G_r^{\star})=\frac{d}{dr}\left(\frac{1}{2}r\right)=1/2$$

5.5. **Defining Equivalent and Non-Equivalent families of Bounded Functions.** Let $n\in\mathbb{N}$ and suppose $f:A\subseteq\mathbb{R}^n\to\mathbb{R}$ is a function, where A and f is Borel:

1. $\{f_r : r\in\mathcal{A}(A)\}$ (Section 2.3.1) is a family of functions, where $\{A_r : r\in\mathcal{A}(A)\}$ is a family of bounded sets and $f_r : A_r\to\mathbb{R}$ is a bounded function
2. $\{g_v : v\in\mathcal{A}(A)\}$ (Section 2.3.1) is a family of functions, where $\{B_v : v\in\mathcal{A}(A)\}$ is a family of bounded sets and $g_v : B_v\to\mathbb{R}$ is a bounded function
3. $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension
4. $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra

Definition 1 (**Equivelant families of Bounded Functions**). *If $\{f_r : r\in\mathcal{A}(A)\}$ (Section 2.3.1) and $\{g_v : v\in\mathcal{A}(A)\}$ are equivalent, then for all $f\in\mathbb{R}^A$, where $(f_r,A_r),(g_v,B_v)\to(f,A)$ (Section 2.3.2):*

$$\mathbb{E}[f_r]=\mathbb{E}[g_v]\quad \textit{(Section 2.3.3)}$$

The following definition is a shortcut of Definition 1,

Definition 2 (**Equivelant families of Bounded Functions (Analytical Version)**). *The families of bounded functions $\{f_r : r\in\mathcal{A}(A)\}$ (Section 2.3.1) and $\{g_v : v\in\mathcal{A}(A)\}$ are equivalent, if there exists $N'\in\mathcal{A}(A)$, where for all $r\geq N'$, there exists $v\in\mathcal{A}(A)$, such that:*

$$\mathcal{H}^{dim_H(A_r)}(\{(x_1,\cdots,x_n):(x_1,\cdots,x_n)\in A_r\cup B_v,\ f_r(x_1,\cdots,x_n)\neq g_v(x_1,\cdots,x_n)\})=0$$

and for all $v\geq N'$, there exists $r\in\mathcal{A}(A)$, such that:

$$\mathcal{H}^{dim_H(B_v)}(\{(x_1,\cdots,x_n):(x_1,\cdots,x_n)\in A_r\cup B_v,\ f_r(x_1,\cdots,x_n)\neq g_v(x_1,\cdots,x_n)\})=0$$

Since this definition is over-sophisticated, consider the following example to find a simpler definition:

5.5.1. *Example of Equivalent Families of Bounded Functions.* Suppose:

- $f:\mathbb{R}\to\mathbb{R}$ is a function and $f(x)=x$
- $\{f_r : r\in\mathcal{A}(\mathbb{R}):=\mathbb{R}^+\}$ (Section 2.3.1) is a family, where $\{A_r : r\in\mathcal{A}(\mathbb{R}):=\mathbb{R}^+\}=\{[-r-2,r+2] : r\in\mathbb{R}^+\}$ and $f_r : A_r\to\mathbb{R}$ is a function, such that $f_r(x)=f(x)$ for all $x\in A_r$
- $\{g_v : v\in\mathcal{A}(\mathbb{R}):=\mathbb{R}^+\}$ (Section 2.3.1) is a family, where $\{B_v : v\in\mathcal{A}(\mathbb{R}):=\mathbb{R}^+\}=\{[-v,v]\cup(\mathbb{Q}\cap[-v-1,v+1]) : v\in\mathbb{R}^+\}$ and $g_v : B_v\to\mathbb{R}$ is a function, such that $g_v(x)=f(x)$ for all $x\in B_v$

Now, suppose $N'=3$. Thus, using Definition 2 and the fact that $f_r(x)=f(x)$ for all $x\in A_r$ and $g_v(x)=f(x)$ for all $x\in B_v$, we prove:

(1) For all $r\geq N'=3$, there exists a $5\leq v=: r+2\in\mathbb{R}^+$, where:

$$\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(\{x : x\in A_r\cup B_v,\ f_r(x)\neq g_v(x)\})=\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(A_r\Delta B_v)=0 \tag{123}$$

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[10] For a definition of the notation $\overline{\lim_{\varepsilon\to 0}}$, see Section 5.3.2

which is proven with the following:
In eq. 123, since $A_r = [-r-2, r+2]$ is a 1-d interval, $\dim_{\mathrm{H}}(A_r) = 1$. Hence,

$$\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(A_r \Delta B_v) = \tag{124}$$

$$\mathcal{H}^1([-r-2, r+2]\Delta([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1]))) = \tag{125}$$

$$\mathcal{H}^1([-r-2, r+2]\Delta([-(r+2), (r+2)] \cup (\mathbb{Q} \cap [-(r+2)-1, (r+2)+1])) = \tag{126}$$

$$\mathcal{H}^1([-r-2, r+2]\Delta([-(r+2), (r+2)] \cup (\mathbb{Q} \cap [-r-3, r+3])) = \tag{127}$$

$$\mathcal{H}^1(\mathbb{Q} \cap ([r+2, r+3]) \cup (\mathbb{Q} \cap [-r-3, r-2])) = 0 \tag{128}$$

(2) For all $v \geq N' = 3$, there exists a $1 \leq v - 2 =: r \in \mathbb{R}^+$, where:

$$\mathcal{H}^{\dim_{\mathrm{H}}(B_v)}(\{x : x \in A_r \cup B_v,\ f_r(x) \neq g_v(x)\}) = \mathcal{H}^{\dim_{\mathrm{H}}(B_v)}(A_r \Delta B_v) = 0 \tag{129}$$

which is proven with the following:
In eq. 129, since $\dim_{\mathrm{H}}(B_v) = \dim_{\mathrm{H}}([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1])) = 1$:

$$\mathcal{H}^{\dim_{\mathrm{H}}(B_v)}(A_r \Delta B_v) = \tag{130}$$

$$\mathcal{H}^1([-r-2, r+2]\Delta([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1]))) = \tag{131}$$

$$\mathcal{H}^1([-(v-2)-2, (v-2)+2]\Delta([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1])) = \tag{132}$$

$$\mathcal{H}^1([-v, v]\Delta([-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1])) = \tag{133}$$

$$\mathcal{H}^1((\mathbb{Q} \cap [-v-1, -v]) \cup (\mathbb{Q} \cap [v, v+1])) = 0 \tag{134}$$

Since Section 5.5.1 crit. (1) and (2) is true, using Definition 2, we have shown $\{A_r : r \in \mathcal{A}(\mathbb{R}) := \mathbb{R}^+\} = \{[-r-2, r+2] : r \in \mathbb{R}^+\}$ and $\{B_v : v \in \mathcal{A}(\mathbb{R}) := \mathbb{R}^+\} = \{[-v, v] \cup (\mathbb{Q} \cap [-v-1, v+1]) : v \in \mathbb{R}^+\}$ are equivalent.

From the examples in Section 5.5.1, we can simplify Definition 2.

Definition 3 (**Definition of Equivalent Families of Bounded Functions (Simplified)**). *Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function. Suppose $f_r : A_r \to \mathbb{R}$ and $g_v : B_v \to \mathbb{R}$ are bounded functions, such that $f_r(x) = f(x)$ for all $x \in A_r$ and $g_v(x) = f(x)$ for all $x \in B_v$.*

The families of bounded functions $\{f_r : r \in \mathcal{A}(A)\}$ and $\{g_v : v \in \mathcal{A}(A)\}$ are equivalent, if there exists $N' \in \mathcal{A}(A)$, where for all $r \geq N'$, there exists $v \in \mathcal{A}(A)$, such that:

$$\mathcal{H}^{dim_H(A_r)}(A_r \Delta B_v) = 0$$

and for all $v \geq N'$, there exists $r \in \mathcal{A}(A)$, such that:

$$\mathcal{H}^{dim_H(B_v)}(A_r \Delta B_v) = 0$$

Note 8 (**The Validity of The Leading Question in Section 3.1 Using Equivalent Families of Bounded Functions**). *In the Leading question (Section 3.1), we want all $f_r^\star \in \mathscr{B}$ (Section 3.1.A) such that for all $r, v \in \mathbb{N}$ and $f_r, g_v \in \mathscr{B}$, f_r and g_v are equivalent (Definition 1, 2, and 3).*

Note that when $\{f_r : r \in \mathcal{A}(A)\}$ (Section 2.3.1) and $\{g_v : v \in \mathcal{A}(A)\}$ are equivalent, there does not exist a $f \in \mathbb{R}^A$, where $(f_r, A_r), (g_v, B_v) \to A$ (Section 2.3.2) and $\mathbb{E}[f_r] \neq \mathbb{E}[g_v]$ (Section 2.3.3). The negation of the former statement is the definition of non-equivalent families of bounded sets in Theorem 1. However, we take the negation of Definition 2.

5.5.2. *Definition of Non-Equivalent families of Bounded Functions.*

Definition 4 (**Non-Equivalent families of Bounded Sets**). *If $\{f_r : r \in \mathcal{A}(A)\}$ and $\{g_v : v \in \mathcal{A}(A)\}$ are non-equivalent, there exists a $f \in \mathbb{R}^A$, where $(f_r, A_r), (g_v, B_v) \to (f, A)$ (Section 2.3.2):*

$$\mathbb{E}[f_r] \neq \mathbb{E}[g_v] \quad \textit{(Section 2.3.3)}$$

Definition 5 (**Non-Equivalent families of Bounded Sets (Analytical Version)**). *The families of sets $\{f_r : r \in \mathcal{A}(A)\}$ and $\{g_v : v \in \mathcal{A}(A)\}$ are non-equivalent, if there exists $N' \in \mathcal{A}(A)$, where for all $r \geq N'$, there is either a $v \in \mathcal{A}(A)$, such that:*

$$\mathcal{H}^{dim_H(A_r)}(\{(x_1, \cdots, x_n) : (x_1, \cdots, x_n) \in A_r \cup B_v,\ f_r(x_1, \cdots, x_n) \neq g_v(x_1, \cdots, x_n)\}) \neq 0$$

or for all $v \geq N'$, there exists a $r \in \mathcal{A}(A)$, such that:

$$\mathcal{H}^{dim_H(B_v)}(\{(x_1, \cdots, x_n) : (x_1, \cdots, x_n) \in A_r \cup B_v,\ f_r(x_1, \cdots, x_n) \neq g_v(x_1, \cdots, x_n)\}) \neq 0$$

Therefore, consider the following:

5.5.3. *Example 1 of Non-Equivalent Families of Bounded Sets.* Suppose:

- $f : \mathbb{R} \to \mathbb{R}$ is a function and $f(x) = x$
- $\{f_r : r \in \mathcal{A}(A) := \mathbb{R}^+\}$ (Section 2.3.1) is a family, where $\{A_r : r \in \mathcal{A}(A)\} = \{[-r, r] : r \in \mathbb{R}^+\}$ and $f_r : A_r \to \mathbb{R}$ is a function, such that $f_r(x) = f(x)$ for all $x \in A_r$
- $(g_v)_{r\in\mathbb{N}}$ is a family (Section 2.3.1), where $\{B_v : v \in \mathcal{A}(\mathbb{R}) := \mathbb{R}^+\} = \{[-v, v] : v \in \mathbb{R}^+\}$ and $g_v : B_v \to \mathbb{R}$ is a function, such that $g_v(x) = x + (1/v)\sin(x)$

Now, suppose $N := 1$. Thus, using def. 5, we prove:

(1) For all $r \geq N' = 1$, there exists a $1 \leq r =: v \in \mathbb{R}^+$, where:

$$\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(\{x : x \in A_r \cup B_v,\ f_r(x) \neq g_v(x)\}) \neq 0$$

which is proven using the following:
Since $A_r = [-r, r]$ is a 1-d interval, $\dim_{\mathrm{H}}(A_r) = 1$. Therefore:

$$\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(\{x : x \in A_r \cup B_v,\ f_r(x) \neq g_v(x)\}) = \tag{135}$$

$$\mathcal{H}^1(\{x : x \in [-r, r] \cup [-v, v],\ x \neq (1/v)\sin(x)\}) = \tag{136}$$

$$\mathcal{H}^1(\{x : x \in [-r, r] \cup [-r, r],\ x \neq (1/r)\sin(x)\}) = \tag{137}$$

$$\mathcal{H}^1(\{x \neq t\pi : t \in \mathbb{Z}\} \cap [-r, r]) = \tag{138}$$

$$\neq 0 \tag{139}$$

Since Section 5.5.3 crit. (1) is true, using Definition 5, we have shown $\{f_r : r \in \mathcal{A}(A)\}$ and $\{g_v : v \in \mathcal{A}(A)\}$ are non-equivalent.

5.5.4. *Example 2 of Non-Equivalent families of Bounded Sets.* Suppose:

- $f : \mathbb{R} \to \mathbb{R}$ is a function and $f(x) = x$
- $\{f_r : r \in \mathcal{A}(A)\}$ is a family, where $\{A_r : r \in \mathcal{A}(A) := \mathbb{R}^+\} = \{[-r, r] : r \in \mathbb{R}^+\}$ and $f_r : A_r \to \mathbb{R}$ is a function, such that $f_r(x) = f(x)$ for all $x \in A_r$
- $(g_v)_{r\in\mathbb{N}}$ is a family, where $\{B_v : v \in \mathcal{A}(\mathbb{R}) := \mathbb{R}^+\} = \{[-2v, 2v] : v \in \mathbb{R}^+\}$ and $g_v : B_v \to \mathbb{R}$ is a function, such that $g_v(x) = f(x)$ for all $x \in B_v$

Now, suppose $N := 1$. Thus, using def. 5 and the fact that $f_r(x) = f(x)$ for all $x \in A_r$ and $g_v(x) = f(x)$ for all $x \in B_v$, we prove:

(1) For all $v \geq N' = 1$, there exists a $3 \leq 2v + 1 =: r \in \mathbb{R}^+$, where:

$$\mathcal{H}^{\dim_{\mathrm{H}}(B_v)}(\{x : x \in A_r \cup B_v,\ f_r(x) \neq g_v(x)\}) = \mathcal{H}^{\dim_{\mathrm{H}}(B_v)}(A_r \Delta B_v) \neq 0$$

which is proven using the following:
Since $B_v = [-2v, 2v]$ is a 1-d interval, $\dim_{\mathrm{H}}(B_v) = 1$. Therefore:

$$\mathcal{H}^{\dim_{\mathrm{H}}(B_v)}(A_r \Delta B_v) = \tag{140}$$

$$\mathcal{H}^1([-(2v + 1), 2v + 1]\Delta[-2v, 2v]) = \tag{141}$$

$$\mathcal{H}^1([-2v - 1, 2v + 1]\Delta[-2v, 2v]) = \tag{142}$$

$$\mathcal{H}^1([-2v - 1, 2v + 1]\Delta[-2v, 2v]) = \tag{143}$$

$$\mathcal{H}^1([-2v - 1, -2v] \cup [2v, 2v + 1]) = \tag{144}$$

$$1 + 1 \neq 0 \tag{145}$$

Since Section 5.5.4 crit. (1) is true, using Definition 5, we have shown $\{f_r : r \in \mathbb{R}^+\}$ and $\{g_v : v \in \mathbb{R}^+\}$ are non-equivalent.

5.6. **Reminder.** See if Section 3.1 is easier to understand.

6. ATTEMPT AT ANSWERING THE APPROACH OF SECTION 2.5.1

6.1. **Choice Function.** Suppose we define the following:

1. Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n$ is a function, where A and f are Borel
2. If $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$ (Section 2.3.4) is an arbitrary set, where for all $r \in \mathbb{N}$, $A_r^\star \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r^\star)$ (Section 2.3.4), $f_r^\star \in \mathscr{B}$ satisfies (1), (2), (3), (4) and (5) of the **leading question** in Section 3.1
3. For all $v \in \mathbb{N}$, $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$ and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star}) \cup (\mathfrak{B}(A_r^\star) \setminus \mathscr{B})$
4. $\{G_r^\star : r \in \mathcal{A}(A)\} = \{\text{graph}(f_r^\star) : r \in \mathcal{A}(A)\}$ (Section 2.3.1) is the family of the graph of each $f_r^\star$
5. $\{G_v^{\star\star} : v \in \mathcal{A}(A)\} = \{\text{graph}(f_v^{\star\star}) : v \in \mathcal{A}(A)\}$ (Section 2.3.1) is the family of the graph of each $f_v^{\star\star}$

Further note, from Section 5.3.3 (a), if we take:

$$\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)|} = \tag{146}$$
$$\inf \left\{ \left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \geq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi))) \right\}$$

and from Section 5.3.3 (b), we take:

$$\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)|} = \tag{147}$$
$$\sup \left\{ \left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \leq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi))) \right\}$$

Then, Section 5.3.1 (2), Equation 146, and Equation 147 is:

$$\sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} |\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)| = |\mathcal{S}'(\varepsilon, G_r^\star)| = |\mathcal{S}'| \tag{148}$$

$$\sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)|} = \overline{|\mathcal{S}'(\varepsilon, G_r^\star)|} = \overline{|\mathcal{S}'|} \tag{149}$$

$$\sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)|} = \underline{|\mathcal{S}'(\varepsilon, G_r^\star)|} = \underline{|\mathcal{S}'|} \tag{150}$$

6.2. **Approach.** We manipulate the definitions of Section 5.3.3 (a) and Section 5.3.3 (b) to solve (1), (2), (3), (4) and (5) of the *leading question* in Section 3.1.

6.3. **Potential Answer.**

6.3.1. *Preliminaries (Definition of T).* Suppose $\{G_r^\star : r \in \mathcal{A}(A)\}$ is the family of the graph on each function $f_r^\star$ (Section 2.3.2). Then, whenever

- The average of $G_r^\star$ for every $r \in \mathbb{N}$ is:

$$\text{Avg}(G_r^\star) = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}(G_r^\star)} \int_{G_r^\star} (x_1, \cdots, x_{n+1}) \, d\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)} \tag{151}$$

- $d(P, Q)$ is the $(n+1)$-dimensional Euclidean distance between points $P, Q \in \mathbb{R}^{n+1}$
- The difference of point $X = (x_1, \cdots, x_{n+1})$ and $Y = (y_1, \cdots, y_{n+1})$ is:

$$X - Y = (x_1 - y_1, x_2 - y_2, \cdots, x_{n+1} - y_{n+1})$$

We want an *explicit* injective $\mathscr{F} : \mathbb{R}^n \to \mathbb{R}$, where $r, v \in \mathbb{N}$, such that:

1. If $d(\text{Avg}(G_r^\star), C) < d(\text{Avg}(G_v^{\star\star}), C)$, then $\mathscr{F}(\text{Avg}(G_r^\star) - C) < \mathscr{F}(\text{Avg}(G_v^{\star\star}) - C)$
2. If $d(\text{Avg}(G_r^\star), C) > d(\text{Avg}(G_v^{\star\star}), C)$, then $\mathscr{F}(\text{Avg}(G_r^\star) - C) > \mathscr{F}(\text{Avg}(G_v^{\star\star}) - C)$
3. If $d(\text{Avg}(G_r^\star), C) = d(\text{Avg}(G_v^{\star\star}), C)$, then $\mathscr{F}(\text{Avg}(G_r^\star) - C) \neq \mathscr{F}(\text{Avg}(G_v^{\star\star}) - C)$

where we define:

$$T(C, G_r^\star) = \mathscr{F}(\text{Avg}(G_r^\star) - C) \tag{152}$$

We explain the motive[11] of the definition of T after defining the preliminary choice function in Section 6.3.3 and the choice function in Theorem 9.

6.3.2. *Question.* Does T exist? If so, how do we define it?

---

[11] See for the motive behind T in Section 6.4.2

6.3.3. *Preliminary Choice Function.* Hence, using $|\mathcal{S}'|$, $\overline{|\mathcal{S}'|}$, $\underline{|\mathcal{S}'|}$, $E(r)$ (Section 3.1.E), $\mathcal{E}(C,G_r^{\star})$ (Section 5.4), and $T(C,G_r^{\star})$, such that with the absolute value function $||\cdot||$, ceiling function $\lceil\cdot\rceil$, and nearest integer function $[\cdot]$, we define the *preliminary choice function*:

$$K(\varepsilon,G_r^{\star})=$$

$$(1+||E(r)-\mathcal{E}(C,G_r^{\star})||)\left(\left|\left|\frac{|\mathcal{S}'|\left(1+\left\lceil\frac{|\mathcal{S}'|\left(\underline{|\mathcal{S}'|}+2|\mathcal{S}'|\right)}{\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|\right)\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|+\overline{|\mathcal{S}'|}\right)}\right\rceil\right)\left(1+\left[\underline{|\mathcal{S}'|}/\,|\mathcal{S}'|\right]\right)}{\left(1+\left[|\mathcal{S}'|\,/\overline{|\mathcal{S}'|}\right]\right)\left(1+\left[\underline{|\mathcal{S}'|}/\overline{|\mathcal{S}'|}\right]\right)}-\left|\mathcal{S}'\right|\right|\right|+\left|\mathcal{S}'\right|\right)-T(C,G_r^{\star})\mathcal{E}(C,G_r^{\star}) \tag{153}$$

where $\mathcal{E}$, E, and T are "removed" when $\mathcal{E},E=0$, the choice function which answers the **leading question** in Section 3.1 could be the following:[12]

Theorem 9. *If we define:*

$$\mathcal{M}(\varepsilon,G_r^{\star})=|\mathcal{S}'(\varepsilon,G_r^{\star})|\big(K(\varepsilon,G_r^{\star})-|\mathcal{S}'(\varepsilon,G_r^{\star})|\big) \tag{154}$$

$$\mathcal{M}(\varepsilon,G_v^{\star\star})=|\mathcal{S}'(\varepsilon,G_v^{\star\star})|\big(K(\varepsilon,G_v^{\star\star})-|\mathcal{S}'(\varepsilon,G_v^{\star\star})|\big) \tag{155}$$

where for $\mathcal{M}(\varepsilon,G_r^{\star})$, we define $\mathcal{M}(\varepsilon,G_r^{\star})$ to be the same as $\mathcal{M}(\varepsilon,G_v^{\star\star})$ when swapping "$v\in\mathbb{N}$" with "$r\in\mathbb{N}$" (for Equation 146 & 147) and sets $G_r^{\star}$ with $G_v^{\star\star}$ (for Equation 146–153), then for constant $v>0$ and variable $v^*>0$, if:[13]

$$\overline{\mathcal{S}}(\varepsilon,r,v^*,G_v^{\star\star})=\inf\left(\{|\mathcal{S}'(\varepsilon,G_v^{\star\star})|:v\in\mathbb{N},\mathcal{M}(\varepsilon,G_v^{\star\star})\geq\mathcal{M}(\varepsilon,G_r^{\star})\geq v^*\}\cup\{v^*\}\right)+v \tag{156}$$

and:

$$\underline{\mathcal{S}}(\varepsilon,r,v^*,G_v^{\star\star})=\sup\left(\{|\mathcal{S}'(\varepsilon,G_v^{\star\star})|:v\in\mathbb{N},v^*\leq\mathcal{M}(\varepsilon,G_v^{\star\star})\leq\mathcal{M}(\varepsilon,G_r^{\star})\}\cup\{-v^*\}\right)+v \tag{157}$$

where for all $r,v\in\mathbb{N}$, there exists a $A_r^{\star}\in\mathcal{B}$ and $f_r^{\star}\in\mathscr{B}$ (Section 6.1 crit. 2), such that for all $A_v^{\star\star}\in\mathbf{B}(\mathbb{R}^n)\setminus\mathcal{B}$ and $f_v^{\star\star}\in\mathfrak{B}(A_v^{\star\star})\cup(\mathfrak{B}(A_r^{\star})\setminus\mathscr{B})$ (Section 6.1 crit. 3), whenever:[14]

$$c=\inf\left\{||1-\mathbf{c_1}||:\forall(\epsilon>0)\exists(\mathbf{c_1}>0)\forall(r\in\mathbb{N})\exists(v\in\mathbb{N})\left(\left|\left|\frac{|\mathcal{S}'(\varepsilon,G_r^{\star})|}{|\mathcal{S}'(\varepsilon,G_v^{\star\star})|}-\mathbf{c_1}\right|\right|<\varepsilon\right)\right\}, \tag{158}$$

and:[15]

- $E:\mathcal{A}(A)\to\mathbb{R}$ *is a function and the fixed rate of expansion*
- $\mathrm{E}_1(r)=\begin{cases}E(r) & E(r)>0\\ 1 & E(r)=0\end{cases}$
- $n\in\mathbb{N}$ *is the dimension of* $\mathbb{R}^n$
- $d(X,Y)$ *is the n-dimensional Euclidean distance between points* $X,Y\in\mathbb{R}^n$
- $\mathbf{D}_r=\sup\{d(x,y):x,y\in G_r^{\star}:=\mathrm{graph}(f_r^{\star})\}$
- $\mathbf{D}'_r=\overline{\lim}_{h\to 0}(\mathbf{D}_{r+h}-\mathbf{D}_r)/h$ *is the generalized derivative*[16] *of* $\mathbf{D}_r$
- $\mathfrak{D}_r=\begin{cases}4/(\mathbf{D}'_r) & \mathbf{D}_r\neq 0\\ 1 & \mathbf{D}'_r=0\end{cases}$
- $\mathrm{Vol}(B_n(C,r))$ *is the volume of an n-dimensional ball of radius r centered at reference point* $C\in\mathbb{R}^n$
- $\mathrm{sign}(\cdot)$ *is the sign function*
- $\lceil\cdot\rceil$ *is the ceiling function*
- $\lfloor\cdot\rfloor$ *is the floor function*
- $\dim_{\mathrm{H}}(G_r^{\star})$ *is the Hausdorff dimension of the set* $G_r^{\star}=\mathrm{graph}(f_r^{\star})\subseteq\mathbb{R}^{n+1}$
- $||\cdot||$ *is the absolute value function*
- *The following is shortened for brevity:*
    - $\mathbf{d_1}:=d_1(G_r^{\star})=\dim_{\mathrm{H}}(G_r^{\star})$
    - $\mathbf{d_2}:=d_2(G_r^{\star},n)=||n-\dim_{\mathrm{H}}(G_r^{\star})||$
    - $\mathbf{s}:=\mathrm{s}(G_r^{\star},n)=1-\frac{2}{n}\dim_{\mathrm{H}}(G_r^{\star})$

---

[12] See Section 6.4 for the reason behind choosing the choice function in Theorem 9

[13] See Section 6.4.5 for the motivation of $\overline{\mathcal{S}}(\varepsilon,r,v^*,G_v^{\star\star})$ and $\underline{\mathcal{S}}(\varepsilon,r,v^*,G_v^{\star\star})$ (Equations 156 and 157)

[14] See Section 6.4.5 for the motivation of c (Equation 158)

[15] See Section 6.4.6 for the motivation of $V(\varepsilon,G_r^{\star},n)$ (Equation 159)

[16] For a definition of the notation $\overline{\lim}_{\varepsilon\to 0}$, see Section 5.3.2

  - $\mathbf{t_1} := \operatorname{sign}(\lfloor \mathbf{d}_1/n \rfloor)$
  - $\mathbf{t_2} := \operatorname{sign}(\lceil \mathbf{s} \rceil)$
- $\mathrm{D}_r = \left(\mathbf{D}_{\mathfrak{D}_r \mathrm{E}_1(r)\mathbf{D}_r}\right)^{(\mathbf{d_2}/n)\operatorname{sign}(\mathbf{t_2}+||\mathfrak{D}_r\mathrm{E}_1(r)\mathbf{D}_r||)}$
- $\mathbf{P}(r)$ *is the partial sum formula of* $\prod_{k=1}^{r}(\mathbf{t_1}+k^{\mathbf{s}})$
- $\mathbf{P}_1(r) = \mathbf{P}\left(\mathfrak{D}_r\mathrm{E}_1(r)(\mathbf{D}_r)^{\operatorname{sign}(E(r))} r^{(\mathbf{d_2}/n)(1-\operatorname{sign}(E(r)))}\right)$

then if:[17]

$$V(\varepsilon, G_r^{\star}, n) = \tag{159}$$
$$\left\lceil \left(2^{\mathbf{d_2}(1-\operatorname{sign}(\mathfrak{D}_r E(r)\mathbf{D_r}))}\left(\frac{\operatorname{Vol}(B_{\mathbf{d_1}}(C,1))}{\operatorname{Vol}(B_{\mathbf{d_2}}(C,1))}\right)(\mathbf{t_2}+\mathrm{D}_r\mathbf{P}_1(r))^n\right)/\varepsilon\right\rceil \Big/ |\mathcal{S}'(\varepsilon, G_r^{\star})|$$

the choice function is:[18]

$$\overline{\limsup_{\varepsilon\to 0}}\lim_{v^*\to\infty}\limsup_{r\to\infty}\left(\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_r^{\star}))\overline{\mathcal{S}}(\varepsilon,r,v^*,G_v^{\star\star})}{|\mathcal{S}'(\varepsilon,G_r^{\star})|+v} - c^{-V(\varepsilon,\,G_r^{\star},n)}\right)\right. \tag{160}$$
$$\left.\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_r^{\star}))\underline{\mathcal{S}}(\varepsilon,r,v^*,G_v^{\star\star})}{|\mathcal{S}'(\varepsilon,G_r^{\star})|+v} - c^{-V(\varepsilon,G_r^{\star},n)}\right)\right) =$$

$$\overline{\liminf_{\varepsilon\to 0}}\lim_{v^*\to\infty}\liminf_{r\to\infty}\left(\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_r^{\star}))\overline{\mathcal{S}}(\varepsilon,k,v^*,G_v^{\star\star})}{|\mathcal{S}'(\varepsilon,G_r^{\star})|+v} - c^{-V(\varepsilon,G_r^{\star},n)}\right)\right. \tag{161}$$
$$\left.\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon,G_r^{\star}))\underline{\mathcal{S}}(\varepsilon,r,v^*,G_v^{\star\star})}{|\mathcal{S}'(\varepsilon,G_r^{\star})|+v} - c^{-V(\varepsilon,G_r^{\star},n)}\right)\right) = 0$$

where $\{G_r^{\star} : r \in \mathcal{A}(A)\} := \{\operatorname{graph}(f_r^{\star}) : r \in \mathcal{A}(A)\}$ *satisfies Equation 160 & Equation 161. Therefore, the expected value which answers the leading question (Section 3.1) is* $\mathbb{E}[f_r^{\star}]$*. (Note, we want* $\sup\emptyset = -\infty$ *and* $\inf\emptyset = +\infty$*.)*

6.4. **Explaining The Choice Function and Evidence The Choice Function Is Credible.**

Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function, where A and f are Borel.

We start our reasoning with f which has countably infinite or discrete graphs, then generalize the reasoning to f with an uncountable and non-discrete graph.

6.4.1. *Evidence With Programming.*

Since the choice function in Equations 160 and 161 is long and extremely sophisticated, we use programming to simplify the calculations. Note, it is only necessary to define $\varepsilon > 0$, $n \in \mathbb{N}$, $\dim_{\mathrm{H}}(G_r^{\star})$ (Section 6.1 crit. 2), $|\mathcal{S}'(\varepsilon, G_r^{\star})|$ *and* $|\mathcal{S}'(\varepsilon, G_v^{\star\star})|$ (Equation 148), $E(r)$ *and* $E(v)$ (Section 3.1.E), $\mathcal{E}(C, G_r^{\star})$ *and* $\mathcal{E}(C, G_v^{\star\star})$ (Section 5.4), and $\mathbf{D}_r$ (pg. 27).

Before defining the code, we explain its limitations. (We want to *rewrite* the program.)

Part 1. **Limitations**

The programming in Code 12 produces outputs without errors, when the fixed rate of expansion $E(r) = 0$ and the graph of f is countably infinite. Despite this, `Entropy1` and `Entropy2` cannot use the floor or ceiling function, since solving for `r` or `v` from `Entropy1` and `Entropy2` in `TableLowAlphr`, `TableUpAlphr`, `TableLowAlphv`, and `TableUpAlphv` is beyond Mathematica's scope. The major issue is `q` cannot be large enough, since the computation time of the code would be too long. Hence, when $E(r), E(v) > 0$ or the graph of f is uncountable, we have to rewrite the code or use a supercomputer.

---

[17] See Section 6.4.6 for the motivation of $V(\varepsilon, G_r^{\star}, n)$ (Equation 159)

[18] For the definitions of the notations $\overline{\limsup}_{\varepsilon\to 0}$ and $\overline{\liminf}_{\varepsilon\to 0}$, see Section 5.3.2.

Part 2. Code of Choice Function

Code 12. Code For Equations 160 and 161

```
Clear[”Global‘*”]

eps= (*Value of $\varepsilon$. See Section 5.3.1 step 1*)

(*‘LengthS1’ is $|S'(\varepsilon, G_r^\star)|$ (Equation 149)*)
LengthS1[r_] := LengthS1[r] =

(*‘Entropy1’ is the approximation of $\sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$ using asymptotic analysis
See Equation 5.3.1 crit. 3e*)

Entropy1[r_] := Entropy1[r] =

(*‘LengthS2’ is $|S'(\varepsilon, G_v^{\star\star})|$ (Equation 149)*)
LengthS2[v_] := LengthS2[v] =

(*‘Entropy2’ is the approximation $\sup_{\omega\in\Omega_{\varepsilon,v}} \sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega), \psi)))$ using asmyptotic analysis.
See Equation 5.3.1 crit. 3e*)

Entropy2[v_] := Entropy2[v] =

q = 35; (*We want q to be as large as possible; however, this is limited by
computation time.*)

(*Below is the process of solving 'TableLowAlphr' which is $\underline{|S'(\varepsilon, G_r^\star)|}$. See Equation 148.*)
LowAlphValuesr = Table[
  {sol1[r_] :=
    sol1[r] = Reduce[v > 0 && Entropy2[v] <= Entropy1[r], v, Integers],
   LowSampler = Max[v /. Solve[sol1[r], {v}, Integers]],
   LowAlphr = N[LengthS2[LowSampler]]}, {r, 3, q}];
TableLowAlphr = Table[LowAlphValuesr[[r − 3 + 1, 3]], {r, 3, q}];

(*Below is the process of solving 'TableUpAlphr' which is $\overline{|S'(\varepsilon, G_r^\star)|}$. See Equation 150.*)
UpAlphValuesr = Table[
  {sol11[r_] :=
    sol11[r] =
     Reduce[v < 20 && Entropy2[v] >= Entropy1[r], v, Integers],
   UpSampler = Min[v /. Solve[sol11[r], {v}, Integers]],
   UpAlphr = N[LengthS2[UpSampler]]}, {r, 3, q}];
TableUpAlphr = Table[UpAlphValuesr[[r − 3 + 1, 3]], {r, 3, q}];

(*‘a1[r_]’ is shorthand for $\underline{|S'(\varepsilon, G_r^\star)|}$. See Equation 148.*)
a1[r_] :=
 a1[r] = TableUpAlphr[[r − 3 + 1]];

(*‘b1[r_]’ is shorthand for $|S'(\varepsilon, G_r^\star)|$. See Equation 149.*)
b1[r_] := b1[r] = LengthS1[r];

(*‘c1[r_]’ is shorthand for $\overline{|S'(\varepsilon, G_r)|}$. See Equation 150.*)
c1[r_] := c1[r] = TableLowAlphr[[r − 3 + 1]];

(*Below is the fixed rate of expansion of $G_r^\star$. Note, ‘E_0’=E. For simplicity, ‘E_0’ is a constant. *)
Subscript[E, 0][r_] :=
 Subscript[E, 0][r] =

(*‘E_1’ is on pg. 27*)
E1[r_] :=
 E1[r] = Subscript[E, 0][r] − Sign[Subscript[E, 0][r]] + 1;

(* ‘ActualE1[r_]’ is the actual rate of expansion of $G_r^\star$. See Section 5.4.*)
ActualE1[r_] := ActualE1[r] = 0;

(*‘K1’ is $K(\varepsilon, G_r^\star)$. See Equation 153.*)
K1[r_] :=
 K1[r] = N[(1 +
```

```
        RealAbs[ActualE1[r] -
          Subscript[E, 0][
           r]]) (RealAbs[(b1[
              r] (1 +
               Ceiling[(b1[
                    r] (a1[r] + 2 b1[r]))/((a1[r] + b1[r]) (a1[r] +
                     b1[r] + c1[r]))]) (1 + Round[a1[r]/b1[r]]))/((1 +
               Round[b1[r]/c1[r]]) (1 + Round[a1[r]/c1[r]])) - b1[r]] +
         b1[r]) + (ActualE1[r])];

(*Below is the process of solving 'TableLowAlphv' which is $\underline{|S'(\varepsilon, G_v^{\star\star})|}$. See Equation 148.*)
LowAlphValuesv = Table[
  {sol2[v_] :=
    sol2[v] = Reduce[r > 0 && Entropy1[r] <= Entropy2[v], r, Integers],
   LowSamplev = Max[r /. Solve[sol2[v], {r}, Integers]],
   LowAlphv = N[LengthS1[LowSamplev]]}, {v, 3, q}];
TableLowAlphv = Table[LowAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}];

(*Below is the process of solving 'TableUpAlphv' which is $\overline{|S'(\varepsilon, G_v^{\star\star})|}$. See Equation 150*)
UpAlphValuesv = Table[
   {sol21[v_] :=
     sol21[v] =
      Reduce[r < 20 && Entropy1[r] >= Entropy2[v], r, Integers],
    UpSamplev = Min[r /. Solve[sol21[v], {r}, Integers]],
    UpAlphv = N[LengthS1[UpSamplev]]}, {v, 3, q}];
TableUpAlphv = Table[UpAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}];

(*'a2[v_]' shorthand for $|S'(\varepsilon, G_v^{\star\star})|$. See Equation 148.*)
a2[v_] :=
 a2[v] = TableUpAlphv[[v - 3 + 1]];

 (*'b2[v_]' is shorthand for $|S'(\varepsilon, G_v^{\star\star})|$. See Equation 149.*)
b2[v_] := b2[v] = LengthS2[v];

(*'c2[v_]' shorthand for $|S'(\varepsilon, G_v^{\star\star})|$. See Equation 150.*)
c2[v_] :=
 c2[v] = TableLowAlphv[[v - 3 + 1]];

 (*'ActualE2[r_]' is the actual rate of expansion of $G_r^{\star}$. See Section 5.4.*)
ActualE2[r_] := ActualE2[r] =

(*'K2' is $K(\varepsilon, G_v^{\star\star})$*)
K2[v_] :=
 K2[v] = N[(1 +
        RealAbs[ActualE2[v] -
          Subscript[E, 0][
           v]]) (RealAbs[(b2[
              v] (1 +
               Ceiling[(b2[
                    v] (a2[v] + 2 b2[v]))/((a2[v] + b2[v]) (a2[v] +
                     b2[v] + c2[v]))]) (1 + Round[a2[v]/b2[v]]))/((1 +
               Round[b2[v]/c2[v]]) (1 + Round[a2[v]/c2[v]])) - b2[v]] +
         b2[v]) + (ActualE2[v])];

(*'Mr' is $M(\varepsilon, G_r^{\star})$. See Equation 154*)
Mr = Table[N[LengthS1[r] (K1[r] - LengthS1[r])], {r, 3, q - 1}];

(*'Mv' is $M(\varepsilon, G_v^{\star\star})$. See Equation 155*)
Mv = Table[N[LengthS2[v] (K2[v] - LengthS2[v])], {v, 3, q - 1}];

(*'UpS' is $\overline{S}(\varepsilon, r, v^{*}, G_v^{\star\star})$. See Equation 156. We could not add $v^{\star}$ or $v$ due to limitations
in programming.*)
UpS = Table[
  LengthS2[Flatten[
      Position[Mv, Min[Select[Mv, # >= Mr[[r - 4 + 2]] &]]]][[1]] +
     4 - 2], {r, 4, q - 3}];

(*'DownS' is $\underline{S}(\varepsilon, r, v^{*}, G_v^{\star\star})$. See Equation 157. We could not add $v^{\star}$ or $v$ due to limitations
in programming.*)
DownS = Table[
```

```
  LengthS2[Flatten[
      Position[Mv, Max[Select[Mv, # <= Mr[[r - 4 + 2]] &]]][[1]] +
    4 - 2], {r, 4, q - 3}];

n = (*Dimension n ∈ ℕ of ℝ^n*)
dimH = (*Hausdorff Dimension of G_r^⋆*)

(*'Subscript[D, 0][r_]' is the same as D_r on pg. 27.*)
Subscript[D, 0][r_] := Subscript[D, 0][r] = ;


(*'Df1[r_]' is the same as 𝔇_r on pg. 27*)
Df1[r_] :=
 Piecewise[{{4/(Subscript[D, 0]'[r])^2, Subscript[D, 0]'[r] != 0}, {1,
    Subscript[D, 0]'[r] == 0}}]


(*The functions 'd1', 'd2', 's', 't1' and 't2' can be found in pg. 27.
Due to their complexity, they are treated as constants.*)
d1 = dimH;
d2 = RealAbs[n - dimH];
s = 1 - (2/n) dimH;
t1 = Sign[Floor[d1/n]];
t2 = Sign[s Floor[s]];
VolB[x_] := (Pi^(x/2))/Gamma[x/2 + 1];

(*c[r_] is the simplified version of the constant c in Equation 158*)
c[r_] := c[r] = LengthS1[r]/LengthS2[r];

('V[r]' is V(ε, G_r^⋆, n). See Equation 159)
V[r_] := V[r] =
  Simplify[Ceiling[((2^(d2))^(1 -
          t1 Sign[Subscript[E, 0][r] Df1[r] Subscript[D, 0][
            r]])) (VolB[d1]/
        VolB[d2]) ((t2 + ((Subscript[D, 0][
               E1[r] Df1[r] Subscript[D, 0][r]])^((d2/n) Sign[
                t2 + RealAbs[
                  E1[r] Df1[r] Subscript[D, 0][r]]])) Product[
             t1 + k^s, {k, 1,
              E1[r] Df1[
                r] (Subscript[D, 0][r]^(
                  Sign[Subscript[E, 0][
                    r]])) (r^((d2/n) (1 -
                     Sign[Subscript[E, 0][r]])))}])^n)/eps]/
    LengthS1[r], r > 0]

(*Below is the choice function. See Equations 160 and 161*)
ChoiceFunction =
 Table[N[((Sign[Mr[[r - 5 + 2]]] UpS[[r - 5 + 2]])/(LengthS1[r]) - (c[
          r - 5 + 2])^(-V[r - 5 + 2]))] N[((Sign[
           Mr[[r - 5 + 2]]] DownS[[r - 5 + 2]])/(LengthS1[r]) - (c[
          r - 5 + 2])^(-V[r - 5 + 2]))], {r, 5, q - 3}]
```

6.4.2. *Motivation of T and The Preliminary Choice Function.*

Part 1. Summary

Suppose $E : \mathcal{A}(A) \to \mathbb{R}$ (Section 2.3.1) is a function and the fixed rate of expansion.

When $E > 0$, the function $T(\varepsilon, G_r^\star)$ should have the choice function "choose" between different expected values of families of bounded functions converging to f, where the graphs of the bounded functions in the families are similar (Definition 6) to each other.

Definition 6 (**Similar Families of Bounded Sets**). *The families of sets $\{G_r^\star : r \in \mathcal{A}(A)\}$ and $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$ are similar, if there exists $N' \in \mathcal{A}(A)$ (Section 2.3.1), where for all $r \geq N'$, there exists $v \in \mathcal{A}(A)$, such that:*

$$\mathcal{H}^{dim_H(G_r^\star)}(G_r^\star \Delta G_v^{\star\star}) = 0$$

and for all $v \geq N'$*, there exists* $r \in \mathcal{A}(A)$*, such that:*

$$\mathcal{H}^{dim_H(G_v^{\star\star})}(G_r^\star \Delta G_v^{\star\star}) = 0$$

(Below is the definition of non-similar families of bounded sets.)

Definition 7 (**Non-Similar Families of Bounded Sets**)**.** *The families of sets* $\{G_r^\star : r \in \mathcal{A}(A)\}$ *and* $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$ *are non-similar, if there exists* $N' \in \mathcal{A}(A)$ *(Section 2.3.1), where for all* $r \geq N'$*, there is either a* $v \in \mathcal{A}(A)$*, such that:*

$$\mathcal{H}^{dim_H(G_r^\star)}(G_r^\star \Delta G_v^{\star\star}) \neq 0$$

or for all $v \geq N'$*, there exists a* $r \in \mathcal{A}(A)$*, such that:*

$$\mathcal{H}^{dim_H(G_v^{\star\star})}(G_r^\star \Delta G_v^{\star\star}) \neq 0$$

If the families of the bounded graphs are similar (Definition 6), their expected values have the same “satisfying” (Section 3.1) and finite average.

Thus, in the paragraph before Definition 6, the families of bounded functions should satisfy criteria 1-5 of the leading question (Section 3.1). The hard part is defining a choice function which “chooses” a subset of these families with the same “satisfying” (Section 3.1) and finite expected value.

In addition, the preliminary choice function is based on the following example: a function f with a discrete or countably infinite graph, which is generalized to f with a non-discrete and uncountable graph.

Part 2. Example to Understand T

When f is bounded and the graph of f is discrete or countably infinite, the choice function (Equations 160 and 161) should choose the set of all $f_r^\star \in \mathscr{B}$ (Section 6.1 crit. 2) which are equivalent to each other (Note 8, page 25) and satisfy all the criteria in the leading question (Section 3.1).

For instance, when $A = \mathbb{Q}$ and $f : A \to \mathbb{R}$ is a function (Section 2.2):

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \end{cases} \tag{162}$$

$\mathscr{B}$ should be the set of all equivalent families (Definition 3) of bounded functions to $\{f_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\}$ (Section 6.1 crit. 2) such that:

(1) $f_r^\star : A_r \to \mathbb{R}$ is a function
(2) $E : \mathcal{A}(A) \to \mathbb{R}$ (i.e., $\mathcal{A}(A) = \mathbb{N}$) is a function and the fixed rate of expansion (e.g., $E = 1/2$)
(3) $\{A_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\} = \left(\left\{c/r! : -\frac{1}{2E(r)} r \cdot r! \leq c \leq -\frac{1}{2E(r)} r \cdot r!\right\}\right)_{r \in \mathbb{N}}$
(4) $f_r^\star(x) = f(x)$ for all $x \in A_r^\star$

The reason for the definitions are $(f_r^\star, A_r^\star) \to (f, A)$ (Section 5.1) and the “measure” (Section 5.3.1, Section 5.3.3) of $\{G_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\} = \{\text{graph}(f_r^\star) : r \in \mathbb{N}\}$ increases at a rate superlinear to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = \{\text{graph}(f_v^{\star\star}) : v \in \mathbb{N}\}$, where $(f_v^{\star\star}, A_v^{\star\star}) \to (f, A)$ and $(f_v^{\star\star})_{v \in \mathbb{N}}$ is non-equivalent (Definition 5) to $(f_r^\star)_{r \in \mathbb{N}}$.

Nonetheless, when removing $T(C, G_r^\star)$ from Equation 6.3.3, the choice function in Equations 160 and 161 might be unable to choose unique, “satisfying” (Section 3.1), and finite average of f: e.g., whenever

(5) $R : \mathbb{N} \to \mathbb{R}$ is an arbitrary and non-constant function
(6) $\mathbb{R}^{\mathbb{N}}$ is the set of all $R : \mathbb{N} \to \mathbb{R}$

the choice function (Equations 160 and 161) chooses a family of functions $\{f_r^\star : r \in \mathbb{N}\}$ such that when:

(7) $(A_r^\star(R(r)))_{r \in \mathbb{N}} = (\{c/r! : -R(r) \cdot r! \leq c \leq R(r) \cdot r!\})_{r \in \mathbb{N}}$
(8) $f_r^\star(x) = f(x)$ for all $x \in A_r^\star(R(r))$,

depending on $R_1, R_2 \in \mathbb{R}^{\mathbb{N}}$ (crit. 6), there exists a $f \in \mathbb{R}^A$ where $(f_r^\star, A_r^\star(R_1(r))), (f_v^{\star\star\star}, A_v^{\star\star\star}(R_2(v))) \to (f, A)$ and $\mathbb{E}[f_r^\star] \neq \mathbb{E}[f_v^{\star\star\star}]$. This means without $T(\varepsilon, F_r^\star)$, $\mathbb{E}[f_r^\star]$ could be more than one value and is non-unique.

Hence, we demonstrate the former paragraph with the following code (i.e., Code 13): suppose $R_1(r)$ is `R1[r]` and $R_2(r)$ is `R2[r]`. (Since R_1 and R_2 are non-constant functions, the text in Section 6.4.1 part 1 states we need to rewrite the code or use a supercomputer.) The output of `choicefunction` might diverge to infinity, although the output should converge to zero when `ActualE1[r]=1/R1'[r]`—i.e., $\mathcal{E}(C, G_r^\star) = 1/(\frac{d}{dr}(R_1(r)))$ (Section 5.4). For instance, consider $R_1(r) = 2r$ and $R_2(v) = v$.

Code 13. Code For Non-Constant R_1 and R_2

```
Clear["Global‘*"]
(*You can substitute any value into ‘R2’, ‘LengthS1’, ‘Entropy1’, ‘LengthS2‘, ‘Entropy2’,
‘q’, ‘Subscript[E,0]’, ’ActualRateE1’, ‘ActualRateE2‘ , ’Subscript[D,0]’ *)

eps= (*Value of $\varepsilon$ and is non-constant. See Section 5.3.1 step 1*)

(*‘R1’ is $R_1$*)
R1[r_]:=2r;

(*‘R2’* is $R_2$ and is non-constant)
R2[v_]:=v;

(*‘LengthS1’ is $|S'(\varepsilon, G_r^\star)|$ (Equation 149)*)
LengthS1[r_] := LengthS1[r] = R1[v] v!+1;

(*‘Entropy1’ is the approximation of $\sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$ using asymptotic analysis
See Equation 5.3.1 crit. 3e*)

Entropy1[r_] := Entropy1[r] = Log2[R1[r] r!];

(*‘LengthS2’ is $|S'(\varepsilon, G_v^{\star\star})|$ (Equation 149)*)
LengthS2[v_] := LengthS2[v] = R2[v] v!+1;

(*‘Entropy2’ is the approximation $\sup_{\omega\in\Omega_{\varepsilon,v}} \sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega), \psi)))$ using asmyptotic analysis.
See Equation 5.3.1 crit. 3e*)

Entropy2[v_] := Entropy2[v] = Log2[R2[v] v!];

q = 35; (*We want q to be as large as possible; however, this is limited by
computation time.*)

(*Below is the process of solving ’TableLowAlphr’ which is $\underline{|S'(\varepsilon, G_r^\star)|}$. See Equation 148.*)
LowAlphValuesr = Table[
  {sol1[r_] :=
    sol1[r] = Reduce[v > 0 && Entropy2[v] <= Entropy1[r], v, Integers],
   LowSampler = Max[v /. Solve[sol1[r], {v}, Integers]],
   LowAlphr = N[LengthS2[LowSampler]]}, {r, 3, q}];
TableLowAlphr = Table[LowAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}];

(*Below is the process of solving ‘TableUpAlphr’ which is $\overline{|S'(\varepsilon, G_r^\star)|}$. See Equation 150.*)
UpAlphValuesr = Table[
  {sol11[r_] :=
    sol11[r] =
     Reduce[v < 20 && Entropy2[v] >= Entropy1[r], v, Integers],
   UpSampler = Min[v /. Solve[sol11[r], {v}, Integers]],
   UpAlphr = N[LengthS2[UpSampler]]}, {r, 3, q}];
TableUpAlphr = Table[UpAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}];

(*‘a1[r_]’ is shorthand for $\underline{|S'(\varepsilon, G_r^\star)|}$. See Equation 148.*)
a1[r_] :=
 a1[r] = TableUpAlphr[[r - 3 + 1]];

(*‘b1[r_]’ is shorthand for $|S'(\varepsilon, G_r^\star)|$. See Equation 149.*)
b1[r_] := b1[r] = LengthS1[r];

(*‘c1[r_]’ is shorthand for $\overline{|S'(\varepsilon, G_r)|}$. See Equation 150.*)
c1[r_] := c1[r] = TableLowAlphr[[r - 3 + 1]];

(*Below is the fixed rate of expansion of $G_r^\star$. Note, ‘E_0’=E. For this example. ‘E0’ is
the constant 1/2.*)
Subscript[E, 0][r_] :=
 Subscript[E, 0][r] = 1/R1'[r];
```

```
(*'E_1' is on pg. 27*)
E1[r_] :=
 E1[r] = Subscript[E, 0][r] - Sign[Subscript[E, 0][r]] + 1;

(* 'ActualE1[r_]' is the actual rate of expansion of $G^\star_r$. See Section 5.4.*)
ActualE1[r_] := ActualE1[r] = 1/R1'[r];

(*'K1' is $K(\varepsilon,G^\star_r)$. See Equation 153.*)
K1[r_] :=
 K1[r] = N[(1 +
        RealAbs[ActualE1[r] -
          Subscript[E, 0][
           r]]) (RealAbs[(b1[
              r] (1 +
               Ceiling[(b1[
                     r] (a1[r] + 2 b1[r]))/((a1[r] + b1[r]) (a1[r] +
                      b1[r] + c1[r]))]) (1 + Round[a1[r]/b1[r]]))/((1 +
               Round[b1[r]/c1[r]]) (1 + Round[a1[r]/c1[r]])) - b1[r]] +
          b1[r]) + (ActualE1[r])];

(*Below is the process of solving 'TableLowAlphv' which is $\underline{|S'(\varepsilon,G^{\star\star}_v)|}$. See Equation 148.*)
LowAlphValuesv = Table[
   {sol2[v_] :=
     sol2[v] = Reduce[r > 0 && Entropy1[r] <= Entropy2[v], r, Integers],
    LowSamplev = Max[r /. Solve[sol2[v], {r}, Integers]],
    LowAlphv = N[LengthS1[LowSamplev]]}, {v, 3, q}];
TableLowAlphv = Table[LowAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}];

(*Below is the process of solving 'TableUpAlphv' which is $\overline{|S'(\varepsilon,G^{\star\star}_v)|}$. See Equation 150*)
UpAlphValuesv = Table[
    {sol21[v_] :=
      sol21[v] =
       Reduce[r < 20 && Entropy1[r] >= Entropy2[v], r, Integers],
     UpSamplev = Min[r /. Solve[sol21[v], {r}, Integers]],
     UpAlphv = N[LengthS1[UpSamplev]]}, {v, 3, q}];
TableUpAlphv = Table[UpAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}];

(*'a2[v_]' shorthand for $|S'(\varepsilon,G^{\star\star}_v)|$. See Equation 148.*)
a2[v_] :=
 a2[v] = TableUpAlphv[[v - 3 + 1]];

 (*'b2[v_]' is shorthand for $|S'(\varepsilon,G^{\star\star}_v)|$. See Equation 149.*)
b2[v_] := b2[v] = LengthS2[v];

(*'c2[v_]' shorthand for $|S'(\varepsilon,G^{\star\star}_v)|$. See Equation 150.*)
c2[v_] :=
 c2[v] = TableLowAlphv[[v - 3 + 1]];

 (* 'ActualE2[r_]' is the actual rate of expansion of $G^\star_r$. See Section 5.4.*)
ActualE2[r_] := ActualE2[r] =1/R2'[r];

(*'K2' is $K(\varepsilon,G^{\star\star}_v)$*)
K2[v_] :=
 K2[v] = N[(1 +
        RealAbs[ActualE2[v] -
          Subscript[E, 0][
           v]]) (RealAbs[(b2[
              v] (1 +
               Ceiling[(b2[
                     v] (a2[v] + 2 b2[v]))/((a2[v] + b2[v]) (a2[v] +
                      b2[v] + c2[v]))]) (1 + Round[a2[v]/b2[v]]))/((1 +
               Round[b2[v]/c2[v]]) (1 + Round[a2[v]/c2[v]])) - b2[v]] +
          b2[v]) + (ActualE2[v])];

(*'Mr' is $M(\varepsilon,G^\star_r)$. See Equation 154*)
Mr = Table[N[LengthS1[r] (K1[r] - LengthS1[r])], {r, 3, q - 1}];

(*'Mv' is $M(\varepsilon,G^{\star\star}_v)$. See Equation 155*)
Mv = Table[N[LengthS2[v] (K2[v] - LengthS2[v])], {v, 3, q - 1}];
```

```
(* 'UpS' is $\overline{S}(\varepsilon,r,v^*,G_v^{\star\star})$. See Equation 156. We could not add $v^\star$ or $v$ due to limitations
in programming. *)
UpS = Table[
  LengthS2[Flatten[
      Position[Mv, Min[Select[Mv, # >= Mr[[r - 4 + 2]] &]]][[1]] +
    4 - 2], {r, 4, q - 3}];

(* 'DownS' is $\underline{S}(\varepsilon,r,v^*,G_v^{\star\star})$. See Equation 157. We could not add $v^\star$ or $v$ due to limitations
in programming. *)
DownS = Table[
  LengthS2[Flatten[
      Position[Mv, Max[Select[Mv, # <= Mr[[r - 4 + 2]] &]]][[1]] +
    4 - 2], {r, 4, q - 3}];

n = 1; (*Dimension $n \in \mathbb{N}$ of $\mathbb{R}^n$ *)
dimH = 0; (*Hausdorff Dimension of $G_r^\star$ *)

(* 'Subscript[D, 0][r_]' is the same as $\mathbf{D}_r$ on pg. 27.*)
Subscript[D, 0][r_] := Subscript[D, 0][r] = ;

(* 'Df1[r_]' is the same as $\mathfrak{D}_r$ on pg. 27*)
Df1[r_] :=
 Piecewise[{{4/(Subscript[D, 0]'[r])^2, Subscript[D, 0]'[r] != 0}, {1,
     Subscript[D, 0]'[r] == 0}}]


(*The functions 'd1', 'd2', 's', 't1' and 't2' can be found in pg. 27.
Due to their complexity, they are treated as constants*)
d1 = dimH;
d2 = RealAbs[n - dimH];
s = 1 - (2/n) dimH;
t1 = Sign[Floor[d1/n]];
t2 = Sign[s Floor[s]];
VolB[x_] := (Pi^(x/2))/Gamma[x/2 + 1];

(*c[r_] is the simplified version of the constant c in Equation 158*)
c[r_] := c[r] = LengthS1[r]/LengthS2[r];

('V[r]' is $V(\varepsilon, G_r^\star, n)$. See Equation 159)
V[r_] := V[r] =
  Simplify[Ceiling[((2^(d2))^(1 -
          t1 Sign[Subscript[E, 0][r] Df1[r] Subscript[D, 0][
             r]])) (VolB[d1]/
        VolB[d2]) ((t2 + ((Subscript[D, 0][
                E1[r] Df1[r] Subscript[D, 0][r]])^((d2/n) Sign[
                 t2 + RealAbs[
                   E1[r] Df1[r] Subscript[D, 0][r]]])) Product[
             t1 + k^s, {k, 1,
              E1[r] Df1[
                r] (Subscript[D, 0][r]^(
                  Sign[Subscript[E, 0][
                    r]]) (r^((d2/n) (1 -
                    Sign[Subscript[E, 0][r]]))))}])^n)/eps]/
    LengthS1[r], r > 0]

(*Below is the choice function. See Equations 160 and 161*)
ChoiceFunction =
 Table[N[((Sign[Mr[[r - 5 + 2]]] UpS[[r - 5 + 2]])/(LengthS1[r]) - (c[
          r - 5 + 2])^(-V[r - 5 + 2]))] N[((Sign[
           Mr[[r - 5 + 2]]] DownS[[r - 5 + 2]])/(LengthS1[r]) - (c[
          r - 5 + 2])^(-V[r - 5 + 2]))], {r, 5, q - 3}]
```

Part 3. Example To Understand Preliminary Choice Function

The preliminary choice function is essential to defining a choice function. When f is bounded, the preliminary choice function $K(\varepsilon, G_r^\star)$ should have the final choice function "choose" between families of bounded functions converging to f whose graphs have "measures" increasing at rates superlinear to that of non-equivalent (Definition 5) such families. In other terms, when $|\cdot|$ is the cardinality, we want $\mathscr{B}$ and $f_r^\star \in \mathscr{B}$ (Section 6.1 crit. 2) such that:

(1) $K(\varepsilon, G_r^\star) = |S(\varepsilon, G_r^\star)|$ and $K(\varepsilon, G_r^\star) - |S(\varepsilon, G_r^\star)| = 0$ (Equations 149 and 153)
(2) $K(\varepsilon, G_v^{\star\star}) \neq |S(\varepsilon, G_v^{\star\star})|$ and $K(\varepsilon, G_v^{\star\star}) - |S(\varepsilon, G_v^{\star\star})| \neq 0$ (Equations 149 and 153)

Consider the following example:

Suppose $f : A \to \mathbb{R}$ is a function, where $A = \mathbb{Q} \cap [0, 1]$ and

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0, 1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0, 1] \end{cases} \tag{163}$$

In addition, suppose $(f_r^\star, A_r^\star), (f_v^{\star\star}, A_v^{\star\star}) \to (f, A)$. Then, consider:

(1) $\{A_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\} = (\{\frac{c}{r!} : 1 \leq c \leq r!\})_{r \in \mathbb{N}}$
(2) $\{A_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\{\frac{c}{d} : 1 \leq c \leq d,\ d \leq v\})_{v \in \mathbb{N}}$
(3) $f_r^\star(x) = f(x)$ for all $x \in A_r^\star$,
(4) $f_v^{\star\star}(x) = f(x)$ for all $x \in A_v^{\star\star}$
(5) $\{G_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\} = (\text{graph}(f_r^\star))_{r \in \mathbb{N}}$
(6) $\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\text{graph}(f_v^{\star\star}))_{v \in \mathbb{N}}$
(7) $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension
(8) $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra

When $|\cdot|$ is the cardinality, $|S'(\varepsilon, G_r^\star)| = \lceil \mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}(G_r^\star)/\varepsilon \rceil$ and $|S'(\varepsilon, G_v^{\star\star})| = \lceil \mathcal{H}^{\dim_{\mathrm{H}}(G_v^{\star\star})}(G_v^{\star\star})/\varepsilon \rceil$. Since the graph of f is countably infinite, the smallest $\varepsilon > 0$ can be is 1. Therefore, $\varepsilon = 1$. (When the graph of f is uncountable, $\varepsilon > 0$ should be smaller and approach zero). Hence, $|S'(\varepsilon, G_r^\star)| = \lceil \mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}(G_r^\star)/\varepsilon \rceil \sim \lceil (r!+1)/1 \rceil = \lceil r!+1 \rceil$ and $|S'(\varepsilon, G_v^{\star\star})| = \lceil \phi(v)/1 \rceil \sim \frac{3}{\pi^2}v^2$ where $\phi(v)$ is Euler's Totient function[19].

Using Code 12, we get `LengthS1` is `r!+1` and `LengthS2` is approximately `(3/(Pi^2))v^2` (i.e., the approximation of the Totient function[19]).

Moreover, using Equation 50, $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi))) \sim \log_2(r!)$ and using [18], $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi))) \sim 2\log_2(v) + 1 - \log_2(3\pi)$. Hence, `Entropy1[r]=Log2[r!]` and `Entropy2[v]=2log2[v]+1-log2[3 Pi]`.

Code 14. First Example of Preliminary Choice Function `K[r]`

```
Clear["Global`*"]

eps=1; (*Value of $\varepsilon$ since the graph of f is countably infinite. See Section 5.3.1 step 1*)

(*`LengthS1' is $|S'(\varepsilon,G_r^\star)|$ (Equation 149)*)
LengthS1[r_] := LengthS1[r] = r!+1;

(*`Entropy1' is the approximation of $\sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)))$ using asymptotic analysis
See Equation 5.3.1 crit. 3e*)

Entropy1[r_] := Entropy1[r] = Log2[r!];

(*`LengthS2' is $|S'(\varepsilon,G_v^{\star\star})|$ (Equation 149)*)
LengthS2[v_] := LengthS2[v] = (3/(Pi)^2)v^2;

(*`Entropy2' is the approximation $\sup_{\omega\in\Omega_{\varepsilon,v}} \sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega),\psi)))$ using asmyptotic analysis.
See Equation 5.3.1 crit. 3e*)

Entropy2[v_] := Entropy2[v] = 2Log2[v]+1-Log2[3 Pi];

q = 10; (*We want q to be as large as possible; however, this is limited by
computation time.*)

(*Below is the process of solving `TableLowAlphr' which is $\underline{|S'(\varepsilon,G_r^\star)|}$. See Equation 148.*)
LowAlphValuesr = Table[
  {sol1[r_] :=
     sol1[r] = Reduce[v > 0 && Entropy2[v] <= Entropy1[r], v, Integers],
    LowSampler = Max[v /. Solve[sol1[r], {v}, Integers]],
    LowAlphr = N[LengthS2[LowSampler]]}, {r, 3, q}];
TableLowAlphr = Table[LowAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}];
```

---

[19] The Totient function $\phi(v)$ is the number of positive integers less than positive integer v which are coprime to v

```
(*Below is the process of solving 'TableUpAlphr' which is $\overline{|S'(\varepsilon,G_r^\star)|}$. See Equation 150.*)
UpAlphValuesr = Table[
  {sol11[r_] :=
    sol11[r] =
     Reduce[v < 20 && Entropy2[v] >= Entropy1[r], v, Integers],
   UpSampler = Min[v /. Solve[sol11[r], {v}, Integers]],
   UpAlphr = N[LengthS2[UpSampler]]}, {r, 3, q}];
TableUpAlphr = Table[UpAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}];

(*'a1[r_]' is shorthand for $\underline{|S'(\varepsilon,G_r^\star)|}$. See Equation 148.*)
a1[r_] :=
 a1[r] = TableUpAlphr[[r - 3 + 1]];

(*'b1[r_]' is shorthand for $|S'(\varepsilon,G_r^\star)|$. See Equation 149.*)
b1[r_] := b1[r] = LengthS1[r];

(*'c1[r_]' is shorthand for $\overline{|S'(\varepsilon,G_r)|}$. See Equation 150.*)
c1[r_] := c1[r] = TableLowAlphr[[r - 3 + 1]];

(*Below is the fixed rate of expansion of $G_r^\star$. Note, 'E_0'=E. For simplicity, 'E_0' is a constant. *)
Subscript[E, 0][r_] :=
 Subscript[E, 0][r] = 0; (*We choose zero since we are focusing on functions
 with bounded graphs*)

(*'E_1' is on pg. 27*)
E1[r_] :=
 E1[r] = Subscript[E, 0][r] - Sign[Subscript[E, 0][r]] + 1;

(* 'ActualE1[r_]' is the actual rate of expansion of $G_r^\star$. See Section 5.4.*)
ActualE1[r_] := ActualE1[r] = 0; (*We choose since we are focusing on functions
with bounded graphs*)

(*'K1' is $K(\varepsilon,G_r^\star)$. See Equation 153.*)
K1[r_] :=
 K1[r] = N[(1 +
        RealAbs[ActualE1[r] -
          Subscript[E, 0][
           r]]) (RealAbs[(b1[
              r] (1 +
               Ceiling[(b1[
                     r] (a1[r] + 2 b1[r]))/((a1[r] + b1[r]) (a1[r] +
                      b1[r] + c1[r]))]) (1 + Round[a1[r]/b1[r]]))/((1 +
               Round[b1[r]/c1[r]]) (1 + Round[a1[r]/c1[r]])) - b1[r]] +
         b1[r]) + (ActualE1[r])];

K[5]

(*The output is 121*)
```

The output of `K[5]` is `121`, which is the same as `LengthS1[5]`.

Code 15. Previous Sentence in Code

```
Clear["Global`*"]
r= (*r is a positive integer*)
K[r]-LengthS1[r]
(*The output is zero*)
```

Therefore, `K[r]-LengthS1[r]=0`. Otherwise, we exclude families of bounded functions $f_r^\star \in \mathscr{B}$ (Section 6.1 crit. 2), where $|S(\varepsilon, G_r^\star)| \neq K(\varepsilon, G_r^\star)$ or $|S(\varepsilon, G_v^{\star\star})| - K(\varepsilon, G_v^{\star\star}) = 0$.

For instance, using the function $f : A \to \mathbb{R}$ in Equation 164, where $A = \mathbb{Q} \cap [0, 1]$ and

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \end{cases} \tag{164}$$

when swapping $(G_r^\star)_{r\in\mathbb{N}}$ with $(G_v^{\star\star})_{r\in\mathbb{N}}$ (i.e., swap $f_r^\star$ and $A_r^\star$ with $f_v^{\star\star}$ and $A_v^{\star\star}$ respectively):

1. $\{A_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\} = (\{\frac{c}{d} : 1 \leq c \leq d,\, d \leq r\})_{r\in\mathbb{N}}$
2. $\{A_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\{\frac{c}{v!} : 1 \leq c \leq v!\})_{v\in\mathbb{N}}$

(3) $f_r^\star(x) = f(x)$ for all $x \in A_r^\star$,
(4) $f_v^{\star\star}(x) = f(x)$ for all $x \in A_v^{\star\star}$
(5) $\{G_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\} = (\text{graph}(f_r^\star))_{r\in\mathbb{N}}$
(6) $\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\text{graph}(f_v^{\star\star}))_{v\in\mathbb{N}}$

Note, unlike the former example, $|\mathcal{S}'(\varepsilon, G_r^\star)| = \lceil\phi(r)/\varepsilon\rceil \sim \frac{3}{\pi^2}r^2$ (see footnote[20]) rather than $r! + 1$ and $|\mathcal{S}'(\varepsilon, G_v^{\star\star})| \sim v! + 1$ rather than $\frac{3}{\pi^2}v^2$. In addition, we swap their entropies, where the following code gives us `Unequal[LengthS1[r]-K1[r],0]` and `Unequal[LengthS2[r]-K2[r],0]`.

Code 16. Second Example For Preliminary Choice Function

```
Clear["Global`*"]

eps=1; (*Value of $\varepsilon$ since the graph of f is countably infinite. See Section 5.3.1 step 1*)

(*`LengthS1' is $|S'(\varepsilon,G_r^\star)|$ (Equation 149)*)
LengthS1[r_] := LengthS1[r] = (3/(Pi)^2)r^2;

(*`Entropy1' is the approximation of $\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)))$ using asymptotic analysis
See Equation 5.3.1 crit. 3e*)

Entropy1[r_] := Entropy1[r] = 2Log2[r]+1-Log2[3 Pi];

(*`LengthS2' is $|S'(\varepsilon,G_v^{\star\star})|$ (Equation 149)*)
LengthS2[v_] := LengthS2[v] = v!+1;

(*`Entropy2' is the approximation $\sup_{\omega\in\Omega_{\varepsilon,v}}\sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega),\psi)))$ using asmyptotic analysis.
See Equation 5.3.1 crit. 3e*)

Entropy2[v_] := Entropy2[v] = Log2[v!];


q = 10; (*We want q to be as large as possible; however, this is limited by
computation time.*)

(*Below is the process of solving `TableLowAlphr' which is $\underline{|S'(\varepsilon,G_r^\star)|}$. See Equation 148.*)
LowAlphValuesr = Table[
  {sol1[r_] :=
    sol1[r] = Reduce[v > 0 && Entropy2[v] <= Entropy1[r], v, Integers],
   LowSampler = Max[v /. Solve[sol1[r], {v}, Integers]],
   LowAlphr = N[LengthS2[LowSampler]]}, {r, 3, q}];
TableLowAlphr = Table[LowAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}];

(*Below is the process of solving `TableUpAlphr' which is $\overline{|S'(\varepsilon,G_r^\star)|}$. See Equation 150.*)
UpAlphValuesr = Table[
  {sol11[r_] :=
    sol11[r] =
     Reduce[v < 2000 && Entropy2[v] >= Entropy1[r], v, Integers],
   UpSampler = Min[v /. Solve[sol11[r], {v}, Integers]],
   UpAlphr = N[LengthS2[UpSampler]]}, {r, 3, q}];
TableUpAlphr = Table[UpAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}];

(*`a1[r_]' is shorthand for $\underline{|S'(\varepsilon,G_r^\star)|}$. See Equation 148.*)
a1[r_] :=
 a1[r] = TableUpAlphr[[r - 3 + 1]];

(*`b1[r_]' is shorthand for $|S'(\varepsilon,G_r^\star)|$. See Equation 149.*)
b1[r_] := b1[r] = LengthS1[r];

(*`c1[r_]' is shorthand for $\overline{|S'(\varepsilon,G_r)|}$. See Equation 150.*)
c1[r_] := c1[r] = TableLowAlphr[[r - 3 + 1]];

(*Below is the fixed rate of expansion of $G_r^\star$. Note, `E_0'=E. For simplicity, 'E_0' is a constant. *)
Subscript[E, 0][r_] :=
 Subscript[E, 0][r] = 0;
```

---

[20] The Totient function $\phi(r)$ is the number of positive integers less than positive integer r which are coprime to r

```
(*'E_1' is on pg. 27*)
E1[r_] :=
 E1[r] = Subscript[E, 0][r] - Sign[Subscript[E, 0][r]] + 1;

(* 'ActualE1[r_]' is the actual rate of expansion of $G_r^\star$. See Section 5.4.*)
ActualE1[r_] := ActualE1[r] = 0;

(*'K1' is $K(\varepsilon, G_r^\star)$. See Equation 153.*)
K1[r_] :=
 K1[r] = N[(1 +
        RealAbs[ActualE1[r] -
          Subscript[E, 0][
           r]]) (RealAbs[(b1[
              r] (1 +
               Ceiling[(b1[
                     r] (a1[r] + 2 b1[r]))/((a1[r] + b1[r]) (a1[r] +
                      b1[r] + c1[r]))]) (1 + Round[a1[r]/b1[r]]))/((1 +
               Round[b1[r]/c1[r]]) (1 + Round[a1[r]/c1[r]])) - b1[r]] +
          b1[r]) + (ActualE1[r])];

(*Below is the process of solving 'TableLowAlphv' which is $\underline{|S'(\varepsilon, G_v^{\star\star})|}$. See Equation 148.*)
LowAlphValuesv = Table[
   {sol2[v_] :=
     sol2[v] = Reduce[r > 0 && Entropy1[r] <= Entropy2[v], r, Integers],
    LowSamplev = Max[r /. Solve[sol2[v], {r}, Integers]],
    LowAlphv = N[LengthS1[LowSamplev]]}, {v, 3, q}];
TableLowAlphv = Table[LowAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}];

(*Below is the process of solving 'TableUpAlphv' which is $\overline{|S'(\varepsilon, G_v^{\star\star})|}$. See Equation 150*)
UpAlphValuesv = Table[
   {sol21[v_] :=
     sol21[v] =
      Reduce[r < 20 && Entropy1[r] >= Entropy2[v], r, Integers],
    UpSamplev = Min[r /. Solve[sol21[v], {r}, Integers]],
    UpAlphv = N[LengthS1[UpSamplev]]}, {v, 3, q}];
TableUpAlphv = Table[UpAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}];

(*'a2[v_]' shorthand for $|S'(\varepsilon, G_v^{\star\star})|$. See Equation 148.*)
a2[v_] :=
 a2[v] = TableUpAlphv[[v - 3 + 1]];

 (*'b2[v_]' is shorthand for $|S'(\varepsilon, G_v^{\star\star})|$. See Equation 149.*)
b2[v_] := b2[v] = LengthS2[v];

(*'c2[v_]' shorthand for $|S'(\varepsilon, G_v^{\star\star})|$. See Equation 150.*)
c2[v_] :=
 c2[v] = TableLowAlphv[[v - 3 + 1]];

 (* 'ActualE2[r_]' is the actual rate of expansion of $G_r^\star$. See Section 5.4.*)
ActualE2[r_] := ActualE2[r] =0;

(*'K2' is $K(\varepsilon, G_v^{\star\star})$*)
K2[v_] :=
 K2[v] = N[(1 +
        RealAbs[ActualE[v] -
          Subscript[E, 0][
           v]]) (RealAbs[(b2[
              v] (1 +
               Ceiling[(b2[
                     v] (a2[v] + 2 b2[v]))/((a2[v] + b2[v]) (a2[v] +
                      b2[v] + c2[v]))]) (1 + Round[a2[v]/b2[v]]))/((1 +
               Round[b2[v]/c2[v]]) (1 + Round[a2[v]/c2[v]])) - b2[v]] +
          b2[v]) + (ActualE[v])]


K1[5]-LengthS1[5]
(*The output is 5.06606*)

K2[5]-LengthS2[5]
```

```
(*The output is 30.5*)
```

Hence, in this case (crit. 1-6, page 38), we exclude the example of $f_r^{\star}$ from $\mathscr{B}$ (crit. 2 page 27).

6.4.3. *Motivation of* $\mathcal{M}(\varepsilon, G_r^{\star})$ *and* $\mathcal{M}(\varepsilon, G_v^{\star\star})$. Suppose:

$$\mathcal{M}(\varepsilon, G_r^{\star}) = |\mathcal{S}'(\varepsilon, G_r^{\star})|\big(K(\varepsilon, G_r^{\star}) - |\mathcal{S}'(\varepsilon, G_r^{\star})|\big)$$
$$\mathcal{M}(\varepsilon, G_v^{\star\star}) = |\mathcal{S}'(\varepsilon, G_v^{\star\star})|\big(K(\varepsilon, G_v^{\star\star}) - |\mathcal{S}'(\varepsilon, G_v^{\star\star})|\big).$$

Then, $\lim_{r\to\infty} \mathcal{M}(\varepsilon, G_r^{\star}) = 0$ (i.e., $K(\varepsilon, G_r^{\star}) = |S'(\varepsilon, G_r^{\star})|$) or $\lim_{r\to\infty} \mathcal{M}(\varepsilon, G_r^{\star}) = +\infty$ and $\lim_{v\to\infty} \mathcal{M}(\varepsilon, G_v^{\star\star}) = 0$ (i.e., $K(\varepsilon, G_v^{\star\star}) = |\mathcal{S}'(\varepsilon, G_v^{\star\star})|$) or $\lim_{v\to\infty} \mathcal{M}(\varepsilon, G_v^{\star\star}) = +\infty$.

The purpose of $\mathcal{M}$ (when f is bounded) is to "choose" between different families of bounded function's graphs where their "measures" increase at rates linear to one another. We want $f_r \in \mathscr{B}$ (crit. 2 page 27), such that when $\{G_r^{\star} : r \in \mathcal{A}(A)\} = \{\text{graph}(f_r^{\star}) : r \in \mathcal{A}(A)\}$ and $\{G_r^{\star} : r \in \mathcal{A}(A)\} = \{\text{graph}(f_r^{\star}) : r \in \mathcal{A}(A)\}$, there exists a linear $r_1 : \mathcal{A}(A) \to \mathcal{A}(A)$, where $\mathcal{M}(\varepsilon, G_{r_1(k)}^{\star}) \ll \mathcal{M}(\varepsilon, G_k^{\star\star})$ or $\mathcal{M}(\varepsilon, G_k^{\star}) \ll \mathcal{M}(\varepsilon, G_{r_1(k)}^{\star\star})$.

Note, the former statement is true: i.e., when the "measure" of $\{G_r^{\star} : r \in \mathcal{A}(A)\}$ increases at a rate superlinear or sublinear to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$. Using Section 6.4.2 part 3 on page 36, when the "measure" of $\{G_r^{\star} : r \in \mathcal{A}(A)\}$ increases at a rate superlinear to that of $\{G_v^{\star} : v \in \mathcal{A}(A)\}$, $K(\varepsilon, G_r^{\star}) - |S'(\varepsilon, G_r^{\star})| = 0$ and $\mathcal{M}(\varepsilon, G_r^{\star}) = 0$. Moreover, when the "measure" of $\{G_r^{\star} : r \in \mathcal{A}(A)\}$ increases at a rate sublinear (Section 5.3.5, page 17) to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$, $\mathcal{M}(\varepsilon, G_{r_1(k)}^{\star}) \ll \mathcal{M}(\varepsilon, G_k^{\star\star})$ or $\mathcal{M}(\varepsilon, G_k^{\star}) \ll \mathcal{M}(\varepsilon, G_{r_1(k)}^{\star\star})$.

For instance, in Code 16, the "measure" of $(G_r^{\star})_{r\in\mathbb{N}}$ increases at a rate sublinear to that of $(G_v^{\star\star})_{v\in\mathbb{N}}$: when swapping "$r \in \mathbb{N}$" with $v \in \mathbb{N}$ and $G_r^{\star}$ with $G_v^{\star\star}$, the "measure" of $(G_r^{\star})_{r\in\mathbb{N}}$ increases at a rate superlinear (Section 5.3.4, page 13) to that of $(G_v^{\star\star})_{v\in\mathbb{N}}$. (In Theorem 7, when the "measure" $(G_r^{\star})_{r\in\mathbb{N}}$ increases at a rate superlinear to that of $(G_v^{\star\star})_{v\in\mathbb{N}}$, the "measure" of $(G_v^{\star\star})_{v\in\mathbb{N}}$ increases at a rate sublinear to that of $(G_r^{\star})_{r\in\mathbb{N}}$). In addition, in Code 16, $|\mathcal{S}(\varepsilon, G_k^{\star})| \ll |\mathcal{S}(\varepsilon, G_k^{\star\star})|$ and $K(\varepsilon, G_k^{\star}) - |S'(\varepsilon, G_k^{\star})| \ll K(\varepsilon, G_k^{\star\star}) - |S'(\varepsilon, G_k^{\star\star})|$ for large k:

Code 17. Continuing From Code 16

```
k= (*We want to rewrite Code 16 to allow an output for larger values of ‘k’*);

RatioM1[k_]:=RatioM1[k]=LengthS1[k]/LengthS2[k]
RatioM1[100]
(*The output should be close to zero*)

RatioM2[k_]:=RatioM2[k]=(K1[k]-LengthS1[k])/(K2[k]-LengthS2[k])
RatioM2[100]
(*The output should be close to zero*)
```

Thus, $\mathcal{M}(\varepsilon, G_k^{\star}) \ll \mathcal{M}(\varepsilon, G_k^{\star\star})$ for large k.

However, when the "measure" of $\{G_r^{\star} : r \in \mathcal{A}(A)\}$ increases at a rate linear (Section 5.3.3 crit. 3) to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$, there exists a linear $r_1 : \mathbb{N} \to \mathbb{N}$ such that $\mathcal{M}(\varepsilon, G_k^{\star})$ is proportional to $\mathcal{M}(\varepsilon, G_{r_1(k)}^{\star\star})$ or $\mathcal{M}(\varepsilon, G_{r_1(k)}^{\star})$ is proportional to $\mathcal{M}(\varepsilon, G_k^{\star\star})$ for large k.

For instance, suppose $f : A \to \mathbb{R}$ is a function, where $A = \mathbb{Q} \cap [0, 1]$ and

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \end{cases} \tag{165}$$

such that:

1. $\{A_r^{\star} : r \in \mathcal{A}(A) := \mathbb{N}\} = (\{m/r! : 1 \leq m \leq r!\})_{r\in\mathbb{N}}$
2. $\{A_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = (\{m/(2(v!)) : 1 \leq m \leq 2(v!)\})_{v\in\mathbb{N}}$
3. $f_r^{\star}(x) = f(x)$ for all $x \in A_r^{\star}$
4. $f_v^{\star\star}(x) = f(x)$ for all $x \in A_v^{\star\star}$
5. $\{G_r^{\star} : r \in \mathcal{A}(A) := \mathbb{N}\} = \big(\{(x, f_r^{\star}(x)) : x \in A_r := (\{m/r! : 1 \leq m \leq r!\})_{r\in\mathbb{N}}\big)_{r\in\mathbb{N}}$,
6. $\{G_v^{\star\star} : v \in \mathcal{A}(A) := \mathbb{N}\} = \big(\{(x, f_v^{\star\star}(x)) : x \in A_v := (\{m/(2(v!)) : 1 \leq m \leq 2(v!)\})_{v\in\mathbb{N}}\}\big)_{v\in\mathbb{N}}$
7. $r_1 : \mathbb{N} \to \mathbb{N}$ is linear, $\mathcal{M}(\varepsilon, G_k^{\star})$ is proportional to $\mathcal{M}(\varepsilon, G_{r_1(k)}^{\star\star})$ or $\mathcal{M}(\varepsilon, G_{r_1(k)}^{\star})$ is proportional to $\mathcal{M}(\varepsilon, G_k^{\star\star})$ for large k
8. $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension
9. $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra

when $|\cdot|$ is the cardinality, $|\mathcal{S}'(\varepsilon, G_r^\star)| = \lceil \mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}(G_r^\star)/\varepsilon \rceil$ and $|\mathcal{S}'(\varepsilon, G_v^{\star\star})| = \lceil \mathcal{H}^{\dim_{\mathrm{H}}(G_v^{\star\star})}(G_v^{\star\star})/\varepsilon \rceil$. Since the graph of f is countably infinite, the smallest $\varepsilon > 0$ can be is 1. Therefore, $\varepsilon = 1$. (When the graph of f is uncountable, $\varepsilon > 0$ should be smaller and approach zero). Hence, $|\mathcal{S}'(\varepsilon, G_r^\star)| \sim \lceil (r!+1)/\varepsilon \rceil = \lceil r!+1 \rceil$ and $|\mathcal{S}'(\varepsilon, G_r^\star)| \sim \lceil (2r!+1)/\varepsilon \rceil = \lceil 2r!+1 \rceil$.

Note, using Equation 50, $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_r^\star, \omega), \psi))) \sim \log_2(r!)$ and similarly $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_v^{\star\star}, \omega), \psi))) \sim \log_2(2v!)$. In addition, $r_1(k) = k$, since $|\mathcal{S}'(\varepsilon, G_{r_1(k)}^\star)|/|\mathcal{S}'(\varepsilon, G_k^{\star\star})| = |\mathcal{S}'(\varepsilon, G_k^\star)|/|\mathcal{S}'(\varepsilon, G_k^{\star\star})| \approx 1/2$ is finite for large k.

Hence, `LengthS1[r]=r!+1`, `Entropy1[r]=Log2[r!]`, `LengthS2[v]=2v!+1`, and `Log2[2v!]`. (The output of $\mathcal{M}(\varepsilon, G_r^\star)$ is `Mr` and the output of $\mathcal{M}(\varepsilon, G_v^{\star\star})$ is `Mv`.)

Code 18. Second Example For Preliminary Choice Function

```
Clear["Global`*"]

eps=1; (*Value of $\varepsilon$ since the graph of f is countably infinite. See Section 5.3.1 step 1*)

(*‘LengthS1’ is $|S'(\varepsilon,G_r^\star)|$ (Equation 149)*)
LengthS1[r_] := LengthS1[r] = r!+1

(*‘Entropy1’ is the approximation of $\sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)))$ using asymptotic analysis
See Equation 5.3.1 crit. 3e*)

Entropy1[r_] := Entropy1[r] = Log2[r!]

(*‘LengthS2’ is $|S'(\varepsilon,G_v^{\star\star})|$ (Equation 149)*)
LengthS2[v_] := LengthS2[v] = 2v!+1

(*‘Entropy2’ is the approximation $\sup_{\omega\in\Omega_{\varepsilon,v}} \sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega),\psi)))$ using asmyptotic analysis.
See Equation 5.3.1 crit. 3e*)

Entropy2[v_] := Entropy2[v] = Log2[v!]


q = 10; (*We want q to be as large as possible; however, this is limited by
computation time.*)

(*Below is the process of solving ‘TableLowAlphr’ which is $\underline{|S'(\varepsilon,G_r^\star)|}$. See Equation 148.*)
LowAlphValuesr = Table[
  {sol1[r_] :=
    sol1[r] = Reduce[v > 0 && Entropy2[v] <= Entropy1[r], v, Integers],
   LowSampler = Max[v /. Solve[sol1[r], {v}, Integers]],
   LowAlphr = N[LengthS2[LowSampler]]}, {r, 3, q}];
TableLowAlphr = Table[LowAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}];

(*Below is the process of solving ‘TableUpAlphr’ which is $\overline{|S'(\varepsilon,G_r^\star)|}$. See Equation 150.*)
UpAlphValuesr = Table[
  {sol11[r_] :=
    sol11[r] =
     Reduce[v < 2000 && Entropy2[v] >= Entropy1[r], v, Integers],
   UpSampler = Min[v /. Solve[sol11[r], {v}, Integers]],
   UpAlphr = N[LengthS2[UpSampler]]}, {r, 3, q}];
TableUpAlphr = Table[UpAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}];

(*‘a1[r_]’ is shorthand for $\underline{|S'(\varepsilon,G_r^\star)|}$. See Equation 148.*)
a1[r_] :=
 a1[r] = TableUpAlphr[[r - 3 + 1]];

(*‘b1[r_]’ is shorthand for $|S'(\varepsilon,G_r^\star)|$. See Equation 149.*)
b1[r_] := b1[r] = LengthS1[r];

(*‘c1[r_]’ is shorthand for $\overline{|S'(\varepsilon,G_r)|}$. See Equation 150.*)
c1[r_] := c1[r] = TableLowAlphr[[r - 3 + 1]];

(*Below is the fixed rate of expansion of $G_r^\star$. Note, ‘E_0’=E. For simplicity, ’E_0’ is a constant. *)
Subscript[E, 0][r_] :=
 Subscript[E, 0][r] = 0;
```

```
(*'E_1' is on pg. 27*)
E1[r_] :=
 E1[r] = Subscript[E, 0][r] - Sign[Subscript[E, 0][r]] + 1;

(* 'ActualE1[r_]' is the actual rate of expansion of $G_r^\star$. See Section 5.4.*)
ActualE1[r_] := ActualE1[r] = 0;

(*'K1' is $K(\varepsilon,G_r^\star)$. See Equation 153.*)
K1[r_] :=
 K1[r] = N[(1 +
        RealAbs[ActualE[r] -
          Subscript[E, 0][
           r]]) (RealAbs[(b1[
              r] (1 +
               Ceiling[(b1[
                    r] (a1[r] + 2 b1[r]))/((a1[r] + b1[r]) (a1[r] +
                     b1[r] + c1[r]))]) (1 + Round[a1[r]/b1[r]]))/((1 +
               Round[b1[r]/c1[r]]) (1 + Round[a1[r]/c1[r]])) - b1[r]] +
         b1[r]) + (ActualE[r])];

(*Below is the process of solving 'TableLowAlphv' which is $\underline{|S'(\varepsilon,G_v^{\star\star})|}$. See Equation 148.*)
LowAlphValuesv = Table[
  {sol2[v_] :=
    sol2[v] = Reduce[r > 0 && Entropy1[r] <= Entropy2[v], r, Integers],
   LowSamplev = Max[r /. Solve[sol2[v], {r}, Integers]],
   LowAlphv = N[LengthS1[LowSamplev]]}, {v, 3, q}];
TableLowAlphv = Table[LowAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}];

(*Below is the process of solving 'TableUpAlphv' which is $\overline{|S'(\varepsilon,G_v^{\star\star})|}$. See Equation 150*)
UpAlphValuesv = Table[
   {sol21[v_] :=
     sol21[v] =
      Reduce[r < 20 && Entropy1[r] >= Entropy2[v], r, Integers],
    UpSamplev = Min[r /. Solve[sol21[v], {r}, Integers]],
    UpAlphv = N[LengthS1[UpSamplev]]}, {v, 3, q}];
TableUpAlphv = Table[UpAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}];

(*'a2[v_]' shorthand for $|S'(\varepsilon,G_v^{\star\star})|$. See Equation 148.*)
a2[v_] :=
 a2[v] = TableUpAlphv[[v - 3 + 1]];

 (*'b2[v_]' is shorthand for $|S'(\varepsilon,G_v^{\star\star})|$. See Equation 149.*)
b2[v_] := b2[v] = LengthS2[v];

(*'c2[v_]' shorthand for $|S'(\varepsilon,G_v^{\star\star})|$. See Equation 150.*)
c2[v_] :=
 c2[v] = TableLowAlphv[[v - 3 + 1]];

 (* 'ActualE2[r_]' is the actual rate of expansion of $G_r^\star$. See Section 5.4.*)
ActualE2[r_] := ActualE2[r] =0;

(*'K2' is $K(\varepsilon,G_v^{\star\star})$*)
K2[v_] :=
 K2[v] = N[(1 +
        RealAbs[ActualE[v] -
          Subscript[E, 0][
           v]]) (RealAbs[(b2[
              v] (1 +
               Ceiling[(b2[
                    v] (a2[v] + 2 b2[v]))/((a2[v] + b2[v]) (a2[v] +
                     b2[v] + c2[v]))]) (1 + Round[a2[v]/b2[v]]))/((1 +
               Round[b2[v]/c2[v]]) (1 + Round[a2[v]/c2[v]])) - b2[v]] +
         b2[v]) + (ActualE[v])];

(*'Mr' is $M(\varepsilon,G_r^\star)$. See Equation 154*)
Mr = Table[N[LengthS1[r] (K1[r] - LengthS1[r])], {r, 3, q - 1}];

(*'Mv' is $M(\varepsilon,G_v^{\star\star})$. See Equation 155*)
Mv = Table[N[LengthS2[v] (K2[v] - LengthS2[v])], {v, 3, q - 1}];
```

```
Table[Mr[[r]], {r, 1, q - 3}] (*List of values in {$\mathcal{M}(\varepsilon,G^{\star}_{3}),\mathcal{M}(\varepsilon,G^{\star}_{4}),\cdots,\mathcal{M}(\varepsilon,G^{\star}_{10})$}*)

(*Output: {12.25, 375., 9760.67, 408447., 2.0647*10^7, 1.40901*10^9, 1.15881*10^11}*)

Table[Mj[[j]], {j, 1, q - 3}] (*List of values in {$\mathcal{M}(\varepsilon,G^{\star\star}_{3}),\mathcal{M}(\varepsilon,G^{\star\star}_{4}),\cdots,\mathcal{M}(\varepsilon,G^{\star\star}_{10})$}*)

(*Output: {172.25, 2613.33, 59524.1, 2.24988*10^6, 1.05393*10^8, 7.04491*10^9, 5.50671*10^11}*)

Table[Mr[[k]]/Mj[[k]], {k, 1, q - 3}] (*List of values in {$\mathcal{M}(\varepsilon,G^{\star}_{3})/\mathcal{M}(\varepsilon,G^{\star\star}_{3}),\mathcal{M}(\varepsilon,G^{\star}_{4})/\mathcal{M}(\varepsilon,G^{\star\star}_{4})$,
$\cdots,\mathcal{M}(\varepsilon,G^{\star}_{10})/\mathcal{M}(\varepsilon,G^{\star\star}_{10})$}*)


(*Output: {0.0711176, 0.143495, 0.163978, 0.181541, 0.195905, 0.200004, 0.210435}*)

Differences[
 Table[(Mr[[k]]/Mj[[k]])/Max[Mr[[k]], Mj[[k]]], {k, 1, q - 3}]]

 (*Output: {-0.000357965, -0.0000521539, -2.67413*10^-6, -7.88305*10^-8,-1.83041*10^-9,
 -2.80078*10^-11}*)
```

Thus, using the outputs in Code 13, we assume $\mathcal{M}(\varepsilon, G^{\star}_{r_1(k)})/\mathcal{M}(\varepsilon, G^{\star\star}_{k}) \approx .25$ for large k. Therefore, we must find a way to "choose" between different families of bounded function's graphs where their "measures" increase at rates linear to one another. This is done using $\overline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$ and $\underline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$ (Section 6.4.4).

6.4.4. *Motivation of* $\overline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$ *and* $\underline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$.

Part 1. Motivation for Definitions of $\overline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$ and $\underline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$

Suppose:

$$\overline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v}) = \inf\left(\{|\mathcal{S}'(\varepsilon, G^{\star\star}_{v})| : v \in \mathbb{N}, \mathcal{M}(\varepsilon, G^{\star\star}_{v}) \geq \mathcal{M}(\varepsilon, G^{\star}_{r}) \geq v^{*}\} \cup \{v^{*}\}\right) + v \tag{166}$$

and:

$$\underline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v}) = \sup\left(\{|\mathcal{S}'(\varepsilon, G^{\star\star}_{v})| : v \in \mathbb{N}, v^{*} \leq \mathcal{M}(\varepsilon, G^{\star\star}_{v}) \leq \mathcal{M}(\varepsilon, G^{\star}_{r})\} \cup \{-v^{*}\}\right) + v \tag{167}$$

where $|\mathcal{S}'(\varepsilon, G^{\star}_{k})|$ *and* $|\mathcal{S}'(\varepsilon, G^{\star\star}_{k})|$ are defined in Equation 149, $\mathcal{M}(\varepsilon, G^{\star}_{r})$ *and* $\mathcal{M}(\varepsilon, G^{\star\star}_{v})$ is explained in Section 6.4.3, $v^{\star}$ is a variable, and v is a constant.

Using the Motivation in Section 6.4.3, when the "measure" of $\{G^{\star}_{r} : r \in \mathcal{A}(A)\}$ increases at a rate linear (Section 5.3.3 crit. 3) to that of $\{G^{\star\star}_{v} : v \in \mathcal{A}(A)\}$, there exists a linear $r_1 : \mathbb{N} \to \mathbb{N}$ where $\mathcal{M}(\varepsilon, G^{\star}_{k})$ is proportional to $\mathcal{M}(\varepsilon, G^{\star\star}_{r_1(k)})$ or $\mathcal{M}(\varepsilon, G^{\star}_{r_1(\underline{k})})$ is proportional to $\mathcal{M}(\varepsilon, G^{\star\star}_{k})$ for large k. However, choosing $r_1(k)$—without trial and error—requires defining $\overline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$ (Equation 166) and $\underline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$ (Equation 167).

Despite this, Equations 166 and 167 do not solve the main issue in Section 6.4.3: we want to compare $\mathcal{M}(\varepsilon, G^{\star}_{r})$ and $\mathcal{M}(\varepsilon, G^{\star\star}_{v})$ to choose a linear $r_1 : \mathbb{N} \to \mathbb{N}$, where $\mathcal{M}(\varepsilon, G^{\star}_{k})$ and $\mathcal{M}(\varepsilon, G^{\star\star}_{r_1(k)})$ or $\mathcal{M}(\varepsilon, G^{\star}_{r_1(k)})$ and $\mathcal{M}(\varepsilon, G^{\star\star}_{k})$ are proportional, such that Equations 166 and 167 "picks" a unique function $\mathbf{c} \in \mathbb{N}^{\mathbb{Q}}$ where:

$$\frac{|\mathcal{S}'(\varepsilon, G^{\star}_{k})|}{|\mathcal{S}'(\varepsilon, G^{\star\star}_{r_1(k)})|} \sim \mathbf{c}(1/k) \tag{168}$$

or:

$$\frac{|\mathcal{S}'(\varepsilon, G^{\star}_{r_1(k)})|}{|\mathcal{S}'(\varepsilon, G^{\star\star}_{k})|} \sim \mathbf{c}(k). \tag{169}$$

We also make sure $\overline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$ (Equation 166) and $\underline{\mathcal{S}}(\varepsilon, r, v^{*}, G^{\star\star}_{v})$ (Equation 167) compares $\mathcal{M}(\varepsilon, G^{\star}_{r})$ and $\mathcal{M}(\varepsilon, G^{\star}_{v})$ to choose a linear $r_1 : \mathbb{N} \to \mathbb{N}$, where $\mathcal{M}(\varepsilon, G^{\star}_{k}) \ll \mathcal{M}(\varepsilon, G^{\star\star}_{k})$, such that Equations 166 and 167 "picks" a unique function $\mathbf{c} \in \mathbb{N}^{\mathbb{Q}}$ where:

$$\frac{|\mathcal{S}(\varepsilon, G^{\star}_{k})|}{|\mathcal{S}(\varepsilon, G^{\star\star}_{r_1(k)})|} \sim \mathbf{c}(1/k) := 0 \tag{170}$$

or:

$$\frac{|\mathcal{S}(\varepsilon, G^{\star}_{r_1(k)})|}{|\mathcal{S}(\varepsilon, G^{\star\star}_{k})|} \sim \mathbf{c}(k) := 0. \tag{171}$$

Part 2. **Motivation for $v^\star$ and v**

Note, the reason we add $v^\star$ in $\overline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})$ is when $\{|\mathcal{S}'(\varepsilon, G_v^{\star\star})| : v \in \mathbb{N}, \mathcal{M}(\varepsilon, G_v^{\star\star}) \geq \mathcal{M}(\varepsilon, G_r^\star) \geq v^*\}$ (Equation 166) is empty, then:

$$\inf(\{|\mathcal{S}'(\varepsilon, G_v^{\star\star})| : v \in \mathbb{N}, \mathcal{M}(\varepsilon, G_v^{\star\star}) \geq \mathcal{M}(\varepsilon, G_r^\star) \geq v^*\} \cup \{v^*\}) = v^*$$

rather than $\inf(\emptyset) = +\infty$. More, in $\underline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})$, when $\{|\mathcal{S}'(\varepsilon, G_v^{\star\star})| : v \in \mathbb{N}, v^* \leq \mathcal{M}(\varepsilon, G_v^{\star\star}) \leq \mathcal{M}(\varepsilon, G_r^\star)\}$ is empty, then:

$$\sup(\{|\mathcal{S}'(\varepsilon, G_v^{\star\star})| : v \in \mathbb{N}, v^* \leq \mathcal{M}(\varepsilon, G_v^{\star\star}) \leq \mathcal{M}(\varepsilon, G_r^\star)\} \cup \{-v^*\}) = -v^\star$$

rather than $\sup(\emptyset) = -\infty$. This is important for the next section.

6.4.5. *Motivation of c.* When:

$$c = \inf\left\{ ||1 - \mathbf{c_1}|| : \forall(\epsilon > 0)\exists(\mathbf{c_1} > 0)\forall(r \in \mathbb{N})\exists(v \in \mathbb{N}) \left(\left|\left| \frac{|\mathcal{S}'(\varepsilon, G_r^\star)|}{|\mathcal{S}'(\varepsilon, G_v^{\star\star})|} - \mathbf{c_1} \right|\right| < \varepsilon \right) \right\} \tag{172}$$

Note 10 (Explanation of Equation 172)**.** *To obtain c in Equation 172,* $\mathbf{c_1}$ *must satisfy the following:*

1. $\mathbf{c_1}$ *is positive*
2. $\mathbf{c_1}$ *satisfies* (1) *and the quantified statement in Equation 172*
3. $\mathbf{c_1}$ *satisfies* (1) *and* (2)*, and has the smallest absolute difference from* 1*.*

then suppose:

$$\left\{ \mathcal{J}(r) : r \in \mathbb{N}, \frac{|\mathcal{S}'(\varepsilon, G_r^\star)|}{|\mathcal{S}'(\varepsilon, G_{\mathcal{J}(r)}^{\star\star})|} \sim c \right\}. \tag{173}$$

The choice function of Equation 174 and 175:[21]

$$\overline{\underset{\varepsilon\to 0}{\limsup}} \lim_{v^*\to\infty} \limsup_{r\to\infty} \left(\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\overline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} - c^{-V(\varepsilon,\, G_r^\star,\, n)} \right) \right. \tag{174}$$

$$\left. \left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\underline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} - c^{-V(\varepsilon,\, G_r^\star, n)} \right) \right) =$$

$$\overline{\underset{\varepsilon\to 0}{\liminf}} \lim_{v^*\to\infty} \liminf_{r\to\infty} \left(\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\overline{\mathcal{S}}(\varepsilon, k, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} - c^{-V(\varepsilon,\, G_r^\star, n)} \right) \right. \tag{175}$$

$$\left. \left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\underline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} - c^{-V(\varepsilon, G_r^\star, n)} \right) \right) = 0$$

is true for the following,

1. The "measure" from $\{G_r^\star : r \in \mathcal{A}(A)\}$ increases at a rate superlinear (Section 5.3.3 criteria 1) to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$, since $\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star)) = 0$ (Section 6.4.3) and $c = 0$ (Note 10).

When c exists, the "measure" from $\{G_r^\star : r \in \mathcal{A}(A)\}$ increases at a rate linear (Section 5.3.3 crit. 3) to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\}$. In addition,

2. When $|\mathcal{S}'(\varepsilon, G_r^\star)| < |\mathcal{S}'(\varepsilon, G_{\mathcal{J}(r)}^{\star\star})|$,

$$\overline{\underset{\varepsilon\to 0}{\limsup}} \lim_{v^*\to\infty} \limsup_{r\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\overline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} = 1/c \tag{176}$$

$$\overline{\underset{\varepsilon\to 0}{\liminf}} \lim_{v^*\to\infty} \liminf_{r\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\underline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} = 0 \tag{177}$$

Thus, to satisfy Equation 174 and Equation 175, $-V(\varepsilon, G_r^\star, n) = -1$ and $V(\varepsilon, G_r^\star, n) = 1$.

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[21] For the definitions of the notations $\overline{\underset{\varepsilon\to 0}{\limsup}}$ and $\overline{\underset{\varepsilon\to 0}{\liminf}}$, see Section 5.3.2.

(3) When $|\mathcal{S}'(\varepsilon, G_r^{\star})| > |\mathcal{S}'(\varepsilon, G_{\mathcal{J}(r)}^{\star\star})|$, then:

$$\overline{\limsup_{\varepsilon\to 0}} \lim_{v^*\to\infty} \limsup_{r\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^{\star}))\overline{\mathcal{S}}(\varepsilon, k, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^{\star})| + v} = +\infty \tag{178}$$

$$\overline{\liminf_{\varepsilon\to 0}} \lim_{v^*\to\infty} \liminf_{r\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^{\star}))\underline{\mathcal{S}}(\varepsilon, k, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^{\star})| + v} = 1/c \tag{179}$$

Hence, to satisfy Equation 174 and Equation 175, $-V(\varepsilon, G_r^{\star}, n) = -1$ and $V(\varepsilon, G_r^{\star}, n) = 1$.

6.4.6. *Motivation of* $V(\varepsilon, G_r^{\star}, n)$.

Part 1. Summary

Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel.

Suppose $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n)$ *and* $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star})$ (Section 2.3.4), $(f_v^{\star\star}, A_v^{\star\star}) \to (f, A)$ (Section 2.3.2), $|\mathcal{S}(\varepsilon, G_r^{\star\star\star})|$ (Equation 149), and we define the "measure" (Section 5.3.1, 5.3.3). Hence, whenever:

$$V_1(\varepsilon, G_r^{\star\star\star}, n) = \tag{180}$$

$$\sup_{f\in\mathbb{R}^A}\left(\sup_{A_v^{\star\star}\in\mathbf{B}(\mathbb{R}^n)}\left(\sup_{f_v^{\star\star}\in\mathfrak{B}(A_v^{\star\star}),\,(f_v^{\star\star},A_v^{\star\star})\to(f,A)}\begin{array}{l}(\max\{|\mathcal{S}(\varepsilon, G_r^{\star\star\star}) : A_r^{\star\star\star} \in \mathbf{B}(\mathbb{R}^n),\ f_r^{\star\star\star} \in \mathfrak{B}(A_r^{\star\star\star}),\ (f_r^{\star\star\star}, A_r^{\star\star\star}) \to (f, A) \\ \qquad G_r^{\star\star\star} := \text{graph}(f_r^{\star\star\star}) \text{ increases at a rate linear or superlinear} \\ \qquad \text{to that of } G_v^{\star\star} := \text{graph}(f_v^{\star\star})\})\end{array}\right)\right)$$

then $V(\varepsilon, G_r^{\star}, n)$ is the same as:

$$V(\varepsilon, G_r^{\star}, n) = \lceil V_1(\varepsilon, G_r^{\star\star\star}, n)\rceil / |\mathcal{S}'(\varepsilon, G_r^{\star})| \tag{181}$$

Note, $V(\varepsilon, G_r^{\star}, n)$ is determined with respect to $\dim_{\mathrm{H}}(G_r^{\star})$ (i.e., the Hausdorff dimension of $G_r^{\star}$) and $n \in \mathbb{N}$ (i.e., the dimension of $\mathbb{R}^n$.)

Part 2. Breaking Down $V(\varepsilon, G_r^{\star}, n)$

Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel.

First, note when:[22]

- $E : \mathcal{A}(A) \to \mathbb{R}$ (Section 2.3.1) is a function and the fixed rate of expansion
- $\mathrm{E}_1(r) = \begin{cases} E(r) & E(r) > 0 \\ 1 & E(r) = 0 \end{cases}$
- $n \in \mathbb{N}$ is the dimension of $\mathbb{R}^n$
- $d(X, Y)$ is the n-dimensional Euclidean distance between points $X, Y \in \mathbb{R}^n$
- $\mathbf{D}_r = \sup\{d(x, y) : x, y \in G_r^{\star} := \text{graph}(f_r^{\star})\}$
- $\mathbf{D}_r' = \overline{\lim}_{h\to 0}(\mathbf{D}_{r+h} - \mathbf{D}_r)/h$ is the generalized derivative[23] of $\mathbf{D}_r$
- $\mathfrak{D}_r = \begin{cases} 4/(\mathbf{D}_r')^2 & \mathbf{D}_r' \neq 0 \\ 1 & \mathbf{D}_r' = 0 \end{cases}$
- $\text{Vol}(B_n(C, r))$ is the volume of an n-dimensional ball of radius r centered at reference point $C \in \mathbb{R}^n$
- $\operatorname{sign}(\cdot)$ is the sign function
- $\lceil\cdot\rceil$ is the ceiling function
- $\lfloor\cdot\rfloor$ is the floor function
- $\dim_{\mathrm{H}}(G_r^{\star})$ is the Hausdorff dimension of the set $G_r^{\star} = \text{graph}(f_r^{\star}) \subseteq \mathbb{R}^{n+1}$
- $||\cdot||$ is the absolute value function
- The following is shortened for brevity:
  - $\mathbf{d_1} := d_1(G_r^{\star}) = \dim_{\mathrm{H}}(G_r^{\star})$
  - $\mathbf{d_2} := d_2(G_r^{\star}, n) = ||n - \dim_{\mathrm{H}}(G_r^{\star})||$
  - $\mathbf{s} := \mathrm{s}(G_r^{\star}, n) = 1 - \frac{2}{n}\dim_{\mathrm{H}}(G_r^{\star})$
  - $\mathbf{t_1} := \operatorname{sign}(\lfloor \mathbf{d}_1/n \rfloor)$

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[22] See Note 11 Case 5

[23] For a definition of the notation $\overline{\lim\limits_{\varepsilon\to 0}}$, see Section 5.3.2

  - $\mathbf{t_2} := \operatorname{sign}(\lceil \mathbf{s} \rceil)$
- $\mathrm{D}_r = \left(\mathbf{D}_{\mathfrak{D}_r \mathrm{E}_1(r)\mathbf{D}_r}\right)^{(\mathbf{d_2}/n)\operatorname{sign}(\mathbf{t_2}+||\mathfrak{D}_r\mathrm{E}_1(r)\mathbf{D}_r||)}$
- $\mathbf{P}(r)$ is the partial sum formula of $\prod_{k=1}^{r}\left(\mathbf{t_1}+k^{\mathbf{s}}\right)$
- $\mathbf{P}_1(r) = \mathbf{P}\left(\mathfrak{D}_r\mathrm{E}_1(r)(\mathbf{D}_r)^{\operatorname{sign}(E(r))} r^{(\mathbf{d_2}/n)(1-\operatorname{sign}(E(r)))}\right)$

then when:[24]

$$\begin{aligned}
&V(\varepsilon, G_r^\star, n) = \qquad\qquad (182)\\
&\left\lceil \left(2^{\mathbf{d_2}(1-\operatorname{sign}(\mathfrak{D}_r E(r)\mathbf{D_r}))} \left(\frac{\operatorname{Vol}(B_{\mathbf{d_1}}(C,1))}{\operatorname{Vol}(B_{\mathbf{d_2}}(C,1))}\right) \left(\mathbf{t_2} + \mathrm{D}_r\mathbf{P}_1(r)\right)^n \right) /\varepsilon \right\rceil \Big/ |\mathcal{S}'(\varepsilon, G_r^\star)|
\end{aligned}$$

$V(\varepsilon, G_r^\star, n)$ can be broken down into the following:

$$V(\varepsilon, G_r^\star, n) = \begin{cases} \left\lceil (\operatorname{Vol}(B_n(C,1))(\mathbf{D}_r))^n /\varepsilon \right\rceil / |\mathcal{S}'(\varepsilon, G_r^\star)| & \forall (r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = n) \\ \left\lceil \left(\left(\frac{1}{\operatorname{Vol}(B_n(C,1))} \right) (\mathbf{D}_r r! + 1)^n \right) /\varepsilon \right\rceil / |\mathcal{S}'(\varepsilon, G_r^\star)| & \forall (r \in \mathcal{A}(A))\,(\dim_{\mathrm{H}}(G_r^\star) = 0) \\ \text{See Note 11 Case 5} & \forall (r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) \in [0,n]) \end{cases} \tag{183}$$

Note 11 (Explanation of Equation 183)**.** *Let $n \in \mathbb{N}$ and suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel. Suppose* $\dim_{\mathrm{H}}(\cdot)$ *is the Hausdorff dimension and $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra.*

Note, we explain why we chose the following in Cases 1, 2, 3, and 4.

$$\left\lceil (\operatorname{Vol}(B_n(C,1))(\mathbf{D}_r))^n /\varepsilon \right\rceil / |\mathcal{S}'(\varepsilon, G_r^\star)|$$

and:

$$\left\lceil \left(\left(\frac{1}{\operatorname{Vol}(B_n(C,1))} \right) (\mathbf{D}_r r! + 1)^n \right) /\varepsilon \right\rceil / |\mathcal{S}'(\varepsilon, G_r^\star)|$$

While a simpler formula for $V(\varepsilon, G_r^\star, n)$ exists, using only Equation 183*, there is a reason we use a sophisticated formula for $V(\varepsilon, G_r^\star, n)$ (Case 5).*

In Case 1, 2 and 4*, we take a continuous f, since this is the easiest way to visualize $V(\varepsilon, G_r^\star, n)$ (Equations 180 and 181). However, the function in Case 3 is an exception and should be simple enough to visualize $V(\varepsilon, G_r^\star, n)$.*

1. *Suppose $n \in \mathbb{N}$, $A = \mathbb{R}^n$, and $f : A \to \mathbb{R}$ is a continuous function. Moreover, suppose $E : \mathcal{A}(A) \to \mathbb{R}$ (Section 2.3.1) is a function and the fixed rate of expansion.*
   - (a) *If $E = 0$, then $G = \operatorname{graph}(f)$ is bounded. Hence, $\mathcal{H}^{\dim_{\mathrm{H}}(G)}(G)$ is finite, so $(G_r^\star) = (\operatorname{graph}(f_r^\star))_{r\in\mathbb{N}}$ is a family of bounded sets. In addition, $\{A_r^\star : r \in \mathcal{A}(A)\}$ is a family of bounded sets and $\{f_r^\star : r \in \mathcal{A}(A)\}$ is a family of bounded functions, since $(f_r^\star, A_r^\star) \to (f, A)$ (Section 2.3.2) where $\mathcal{H}^{\dim_{\mathrm{H}}(\mathcal{A}(A))}(\mathcal{A}(A))$ is finite.*
     Therefore, $V(\varepsilon, G_r, n)$ can be any constant, for every $r \in \mathcal{A}(A)$, $n \in \mathbb{N}$, and $\varepsilon > 0$. For the sake of generalization, $V(\varepsilon, G_r^\star, n) = \left\lceil (\operatorname{Vol}(B_n(C,1))(\mathbf{D}_r))^n /\varepsilon \right\rceil / |\mathcal{S}'(\varepsilon, G_r^\star)|$. Notice, $V(\varepsilon, G_r^\star, n)$ is constant, since $\mathbf{D}_r$ is constant (p. 46).
2. *Suppose $n \in \mathbb{N}$, $A = \mathbb{R}^n$, and $f : A \to \mathbb{R}$ is a continuous function. If $E > 0$, then suppose:*
   - (a) *$f_r^\star : A_r \to \mathbb{R}$ is a function*
   - (b) *$\mathbf{D_r}$ is a non-constant indexed family (p. 46)*
   - (c) *$\{A_r^\star : r \in \mathcal{A}(A)\} = \{\operatorname{Vol}(B_n(C, \mathbf{D}_r)) : r \in \mathcal{A}(A)\} = \{\operatorname{Vol}(B_n(C,1))(\mathbf{D}_r)^n : r \in \mathcal{A}(A)\}$*
   - (d) *$f_r^\star(x) = f(x)$ for all $x \in A_r^\star$*
   - (e) *$A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n)$ (Section 2.3.4) and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star})$*
   - (f) *$(f_v^{\star\star}, A_v^{\star\star}) \to (f, A)$ (Section 5.1)*
   - (g) *$\{G_r^\star : r \in \mathcal{A}(A)\} := \{\operatorname{graph}(f_r^\star) : r \in \mathcal{A}(A)\}$*
   - (h) *$\forall (r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = n)$*
   - (i) *Since "the measure" (Section 5.3.1, Section 5.3.3) of $\{G_r^\star : r \in \mathcal{A}(A)\}$ increases at a rate linear and superlinear to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\} = \{\operatorname{graph}(f_v^{\star\star}) : v \in \mathcal{A}(A)\}$ and the actual*

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[24] See Note 11 Case 5

rate of expansion of $\{G_r^\star : r \in \mathcal{A}(A)\}$ (Section 5.4) is $\mathcal{E}(C, G_r^\star) = \mathbf{D}'_r/4$, the largest $|S'(\varepsilon, G_r^\star)|$ (Equation 180) can be is:

$$\left\lceil (\text{Vol}(B_n(C,1))(\mathbf{D}_r))^n \,/\varepsilon \right\rceil$$

Hence, $V(\varepsilon, G_r^\star, n) = \left\lceil (\text{Vol}(B_n(C,1))(\mathbf{D}_r))^n \,/\varepsilon \right\rceil /|\mathcal{S}'(\varepsilon, G_r^\star)|$

(3) When $A = \mathbb{Q}$ and $f : A \to \mathbb{R}$ is a function (Section 2.2):

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap (a,b) \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap (a,b) \end{cases} \tag{184}$$

suppose $E : \mathcal{A}(A) \to \mathbb{R}$ is a function and the fixed rate of expansion where $E = 0$ and:

(a) $f_r^\star : A_r \to \mathbb{R}$ is a function
(b) $\mathbf{D_r} = b - a$ (p. 46)
(c) $\{A_r^\star : r \in \mathcal{A}(A) := \mathbb{N}\} = (\{c/r! : ar! \leq c \leq br!\})_{r\in\mathbb{N}} = (\{c/r! : (b - \mathbf{D}_r)r! \leq c \leq br!\})_{r\in\mathbb{N}}$
(d) $f_r^\star(x) = f(x)$ for all $x \in A_r^\star$
(e) $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n)$ (Section 2.3.4) and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star})$
(f) $(f_v^{\star\star}, A_v^{\star\star}) \to (f, A)$ (Section 5.1)
(g) Since "the measure" (Section 5.3.1, Section 5.3.3) of $(G_r^\star) = (\text{graph}(f_r^\star))_{r\in\mathbb{N}}$ increases at a rate linear and superlinear to that of $(G_v^\star)_{v\in\mathbb{N}} = (\text{graph}(f_v^{\star\star}))_{v\in\mathbb{N}}$, the largest $|S'(\varepsilon, G_r^\star)|$ can be (Equation 180) is:

$$\left\lceil (\mathbf{D}_r r! + 1)/\varepsilon \right\rceil.$$

Hence, $V(\varepsilon, G_r^\star, n) = \left\lceil ((\mathbf{D}_r r! + 1)^n) \,/\varepsilon \right\rceil /|\mathcal{S}'(\varepsilon, G_r^\star)|$

(h) We can generalize (3g) to an arbitrary set A, where $n \in \mathbb{N}$, $\dim_{\text{H}}(A) = 0$, and $A \subseteq \mathbb{R}^n$ such that the following is true:

  (i) $\forall(r \in \mathcal{A}(A))(\dim_{\text{H}}(G_r^\star) = 0)$
  (ii) When $n \in \mathbb{N}$, then "$(\mathbf{D}_r r! + 1)$" in $\left\lceil (\mathbf{D}_r r! + 1)/\varepsilon \right\rceil$ can be extended to $(\mathbf{D}_r r! + 1)^n$
  (iii) When $n \in \mathbb{N}$, we want a "portion" of $(\mathbf{D_r} r! + 1)^n$ for each $r \in \mathcal{A}(A)$ contained in the n-dimensional ball $\text{Vol}(B_n(C, \mathbf{D}_r)) := \text{Vol}(B_n(C,1))(\mathbf{D_r})^n$. (Note, "$\text{Vol}(B_n(C,1))(\mathbf{D_r})^n$" is from the family of n-dimensional balls $\{\text{Vol}(B_n(C,1))(\mathbf{D}_r)^n : r \in \mathcal{A}(A)\}$). Hence, we divide $(\mathbf{D_r} r! + 1)^n$ by $\text{Vol}(B_n(C,1))$: i.e., the largest $|\mathcal{S}'(\varepsilon, G_r^\star)|$ can be is

$$\left\lceil \left(\left(\frac{1}{\text{Vol}(B_n(C,1))}\right)(\mathbf{D}_r r! + 1)^n\right)/\varepsilon \right\rceil$$

Hence, $V(\varepsilon, G_r^\star, n) = \left\lceil \left(\left(\frac{1}{\text{Vol}(B_n(C,1))}\right)(\mathbf{D}_r r! + 1)^n\right)/\varepsilon \right\rceil /\mathcal{S}'(\varepsilon, G_r^\star)$

(4) When $n \in \mathbb{N}$ and $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a continuous function, where A and f is Borel, $\dim_{\text{H}}(A) = 0$, $\forall(r \in \mathcal{A}(A))(\dim_{\text{H}}(G_r^\star) = 0)$ and $E : \mathcal{A}(A) \to \mathbb{R}$ is a function and the fixed rate of expansion such that $E(r) > 0$:

(a) $f_r : A_r^\star \to \mathbb{R}$ is a function
(b) $\mathbf{D}_r$ is a non-constant indexed family (p. 46)
(c) $\{A_r^\star : r \in \mathcal{A}(A)\} = \left\{\left\lceil \frac{1}{\text{Vol}(B_n(C,1))}(\mathbf{D}_r r! + 1)^n \right\rceil : r \in \mathcal{A}(A)\right\}$
(d) $f_r^\star(x) = f(x)$ for all $x \in A_r^\star$
(e) $A_v^\star \in \mathbf{B}(\mathbb{R}^n)$ (Section 2.3.4) and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star})$
(f) $(f_v^{\star\star}, A_v^{\star\star}) \to (f, A)$
(g) Since "the measure" (Section 5.3.1, Section 5.3.3) of $\{G_r^\star : r \in \mathcal{A}(A)\} = \{\text{graph}(f_r^\star) : r \in \mathcal{A}(A)\}$ increases at a rate linear and superlinear to that of $\{G_v^{\star\star} : v \in \mathcal{A}(A)\} = \{\text{graph}(f_v^{\star\star}) : v \in \mathcal{A}(A)\}$ and the actual rate of expansion of $\{G_r^\star : r \in \mathcal{A}(A)\}$ (Section 5.4) is $\mathcal{E}(C, G_r^\star) = \mathbf{D}'_r/4$, the largest $|S'(\varepsilon, G_r^\star)|$ can be (Equation 180) is:

$$\left\lceil \left(\left(\frac{1}{\text{Vol}(B_n(C,1))}\right)(\mathbf{D}_r r! + 1)\right)^n /\varepsilon \right\rceil$$

Hence, $V(\varepsilon, G_r^\star, n) = \left\lceil \left(\left(\frac{1}{\text{Vol}(B_n(C,1))}\right)(\mathbf{D}_r r! + 1)\right)^n /\varepsilon \right\rceil$

(5) For the other cases, besides Case 1, 2, 3, and 4, where the following is shortened for brevity:

- $\mathbf{d_1} := d_1(G_r^\star) = \dim_{\text{H}}(G_r^\star)$
- $\mathbf{d_2} := d_2(G_r^\star, n) = ||n - \dim_{\text{H}}(G_r^\star)||$

- $\mathbf{s} := \mathrm{s}(G_r^\star, n) = 1 - \frac{2}{n}\dim_{\mathrm{H}}(G_r^\star)$
- $\mathbf{t_1} := \mathrm{sign}(\lfloor \mathbf{d_1}/n \rfloor)$
- $\mathbf{t_2} := \mathrm{sign}(\lceil \mathbf{s} \rceil)$

we want a formula generalizing Cases 1-4, based on Equation 180, where we use the following:

(a) *The function and fixed rate of expansion $E : \mathcal{A}(A) \to \mathbb{R}$ (Section 2.3.1)*

(b) $$\mathbf{t_1} = \mathrm{sign}(\lfloor \mathbf{d_1}/n \rfloor) = \begin{cases} 0 & \forall(r \in \mathcal{A}(A))\,(\dim_{\mathrm{H}}(G_r^\star) < n) \\ 1 & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = n) \\ \mathrm{sign}(\lfloor \mathbf{d_1}/n \rfloor) & \textit{Otherwise} \end{cases}$$

(c) $$\mathbf{t_2} = \mathrm{sign}(\lceil s \rceil) = \begin{cases} 1 & \forall(r \in \mathcal{A}(A))(0 \le \dim_{\mathrm{H}}(G_r^\star) < n/2) \\ 0 & \forall(r \in \mathcal{A}(A))(n/2 \le \dim_{\mathrm{H}}(G_r^\star) < n) \\ -1 & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = n) \\ \mathrm{sign}\left(\lceil 1 - \frac{2}{n}\dim_{\mathrm{H}}(G_r^\star) \rceil\right) & \textit{Otherwise} \end{cases}$$

(d) $$\frac{\mathrm{Vol}(B_{\mathbf{d_1}}(C,1))}{\mathrm{Vol}(B_{\mathbf{d_2}}(C,1))} = \begin{cases} \mathrm{Vol}\,(B_n(C,1)) & \mathbf{d_1} = n,\ \mathbf{d_2} = 0 \\ 1/\mathrm{Vol}(B_n(C,1)) & \mathbf{d_1} = 0,\ \mathbf{d_2} = n \\ \mathrm{Vol}(B_{\mathbf{d_1}}(C,1))/\mathrm{Vol}(B_{\mathbf{d_2}}(C,1)) & \textit{Otherwise} \end{cases}$$

(e) $$\mathbf{P}(r) = \prod_{k=1}^r \mathbf{t_1} + k^{\mathbf{s}} = \begin{cases} \prod_{k=1}^r 1 + k^{\mathbf{s}} = r^{\mathbf{s}} & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = n) \ \textit{(Case 1 and 2)} \\ \prod_{k=1}^r k^{\mathbf{s}} = (r!)^{\mathbf{s}} & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = 0) \ \textit{(Case 3 and 4)} \\ \prod_{k=1}^r \mathbf{t_1} + k^{\mathbf{s}} & \textit{Otherwise} \end{cases}$$

(f) $$\mathfrak{D}_r E_1(r)\mathbf{D}_r = \begin{cases} (4/\mathbf{D}'(r)^2)(\mathbf{D}'(r)/4)(\mathbf{D}_r) := \mathbf{D_r}/\mathbf{D'_r} & \mathbf{D}'_r \ne 0 \\ (1)(1)\mathbf{D}_r := \mathbf{D}_r & \mathbf{D}'_r = 0 \end{cases} \ \textit{(p. 46)}$$

(g) $$\mathbf{P}_1(r) = \mathbf{P}(\mathfrak{D}_r \mathrm{E}_1(r)(\mathbf{D}_r)^{\mathrm{sign}(E(r))} r^{(\mathbf{d_2}/n)(1-\mathrm{sign}(E(r)))}) =$$
$$\begin{cases} \mathbf{P}(1) & \forall(r \in \mathcal{A}(A))\,(\dim_{\mathrm{H}}(G_r^\star) = n,\ E(r) = 0) \ \textit{(Case 1)} \\ \mathbf{P}(\mathbf{D}_r/\mathbf{D}'_r) & \forall(r \in \mathcal{A}(A)\,(\dim_{\mathrm{H}}(G_r^\star) = n,\ E(r) > 0) \ \textit{(Case 2)} \\ \mathbf{P}((1/\mathbf{D}'_r)r) & \forall(r \in \mathcal{A}(A))\,(\dim_{\mathrm{H}}(G_r^\star) = 0,\ E(r) = 0) \ \textit{(Case 3)} \\ \mathbf{P}(\mathbf{D}_r) & \forall(r \in \mathcal{A}(A))\,(\dim_{\mathrm{H}}(G_r^\star) = 0,\ E(r) > 0) \ \textit{(Case 4)} \\ \mathbf{P}(\mathfrak{D}_r \mathrm{E}_1(r)(\mathbf{D}_r)^{\mathrm{sign}(E(r))} r^{(\mathbf{d_2}/n)(1-\mathrm{sign}(E(r)))}) & \textit{Otherwise (Case 5)} \end{cases}$$

(h) $$2^{\mathbf{d_2}(1-\mathrm{sign}(\mathfrak{D}_r E_1(r)\mathbf{D}_r)} =$$
$$\begin{cases} 1 & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = n) \ \textit{(Case 1 and 2)} \\ 2^{n(1-\mathrm{sign}(\mathbf{D}_r))} & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = 0,\ E(r) = 0) \ \textit{(Case 3)} \\ 2^{n(1-\mathrm{sign}(\mathbf{D}_r/\mathbf{D}'_r))} & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = 0,\ E(r) > 0) \ \textit{(Case 4)} \\ 2^{\mathbf{d_2}(1-\mathrm{sign}(\mathfrak{D}_r E_1(r)\mathbf{D}_r))} & \textit{Otherwise (Case 5)} \end{cases}$$

(i) $$\mathrm{D}_r = \begin{cases} (\mathbf{D}_{\mathbf{D}_r})^0 := 1 & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = n) \ \textit{(Case 1 and 2)} \\ (\mathbf{D}_{\mathbf{D}_r})^{\mathrm{sign}(1+||\mathbf{D}_r||)} & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = 0,\ E(r) = 0) \ \textit{(Case 3)} \\ (\mathbf{D}_{\mathbf{D}_r/\mathbf{D}'_r})^{\mathrm{sign}(1+||\mathbf{D}_r|)} & \forall(r \in \mathcal{A}(A))(\dim_{\mathrm{H}}(G_r^\star) = 0,\ E(r) > 0) \ \textit{(Case 4)} \\ (\mathbf{D}_{\mathfrak{D}_r \mathrm{E}_1(r)\mathbf{D}_r})^{(\mathbf{d_2}/n)\mathrm{sign}(\mathbf{t_2}+||\mathfrak{D}_r \mathrm{E}_1(r)\mathbf{D}_r||)} & \textit{Otherwise (Case 5)} \end{cases}$$

Hence, we use Case 5 (a)-(i) *to compute:*

$$V(\varepsilon, G_r^\star, n) = \tag{185}$$
$$\left\lceil \left(2^{\mathbf{d_2}(1-\mathrm{sign}(\mathfrak{D}_r E(r)\mathbf{D_r}))} \left(\frac{\mathrm{Vol}(B_{\mathbf{d_1}}(C,1))}{\mathrm{Vol}(B_{\mathbf{d_2}}(C,1))} \right) (\mathbf{t_2} + \mathrm{D}_r \mathbf{P}_1(r))^n \right) / \varepsilon \right\rceil \Big/ |\mathcal{S}'(\varepsilon, G_r^\star)|$$

7. QUESTIONS

1. Does Section 6 answer the **leading question** in Section 3.1
2. Using Theorem 9, when f is defined in Section 2.1, does $\mathbb{E}[f_r^\star]$ have a finite value?
3. Using Theorem 9, when f is defined in Section 2.2, does $\mathbb{E}[f_r^\star]$ have a finite value?
4. If there is no time to check questions 1, 2 and 3, see Section 4.

8. APPENDIX OF SECTION 5.3.1

8.1. Example of Section 5.3.1, step 1.

Suppose

1. $A = \mathbb{R}$
2. When defining $f : A \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \le x < 0.5 \\ 0.5 & 0.5 \le x \end{cases} \tag{186}$$

(3) $\{G^\star_r : r \in \mathcal{A}(A) := \mathbb{R}^+\} = \{\{(x, f^\star_r(x)) : -r \le x \le r\} : r \in \mathcal{A}(A) := \mathbb{R}^+\} =$
$\{\{(x, f(x)) : -r \le x \le r\} : r \in \mathcal{A}(A) := \mathbb{R}^+\}$

Then one example of $\mathbf{C}(\sqrt{2}/6, G^\star_1, 1)$, using Section 5.3.1 step 1, (where $G^\star_1 = \{(x, f(x)) : -1 \le x \le 1\}$) is:

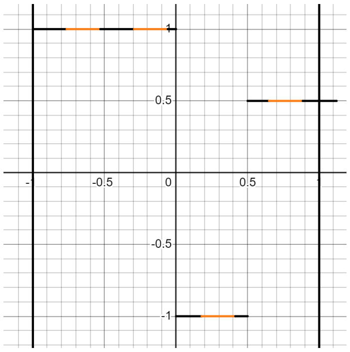
$$\begin{gathered}\left\{\left\{(x,f(x)) : -1 \le x < \frac{\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{\sqrt{2}-6}{6} \le x < \frac{2\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{2\sqrt{2}-6}{6} \le x < \frac{3\sqrt{2}-6}{6}\right\}\right. \\ \left\{(x,f(x)) : \frac{3\sqrt{2}-6}{6} \le x < \frac{4\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{4\sqrt{2}-6}{6} \le x < \frac{5\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{5\sqrt{2}-6}{6} \le x < \frac{6\sqrt{2}-6}{6}\right\} \\ \left.\left\{(x,f(x)) : \frac{6\sqrt{2}-6}{6} \le x < \frac{7\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{7\sqrt{2}-6}{6} \le x < \frac{8\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{8\sqrt{2}-6}{6} \le x \le \frac{9\sqrt{2}-6}{6}\right\}\right\}\end{gathered} \tag{187}$$

Note, the length of each partition is $\sqrt{2}/6$, where the borders could be approximated as:

$$\begin{gathered}\{\,\{(x,f(x)) : -1 \le x < -.764\}, \{(x,f(x)) : -.764 \le x < -.528\}, \{(x,f(x)) : -.528 \le x < -.293\} \\ \{(x,f(x)) : -.293 \le x < -.057\}, \{(x,f(x)) : -.057 \le x < .178\}, \{(x,f(x)) : .178 \le x < .414\} \\ \{(x,f(x)) : .414 \le x < .65\}, \{(x,f(x)) : .65 \le x < .886\}, \{(x,f(x)) : .886 \le x \le 1.121\}\,\}\end{gathered} \tag{188}$$

which is illustrated using *alternating* orange/black lines of equal length covering $G^\star_1$ (i.e., the black vertical lines are the smallest and largest x-cooridinates of $G^\star_1$).

FIGURE 3. The alternating orange *&* black lines are the "covers" and the vertical lines are the boundaries of $G^\star_1$.



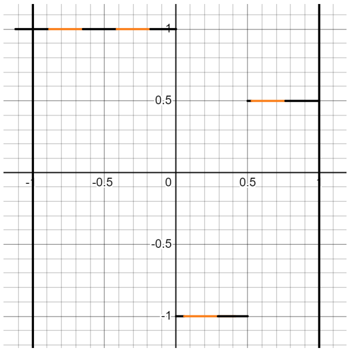
(Note, the alternating covers in Figure 3 satisfy step (1) of Section 5.3.1, because the Hausdorff measure *in its dimension* of the covers is $\sqrt{2}/6$ and there are 9 covers over-covering $G^\star_1$: i.e.,

Definition 8 (**Minimum Covers of Measure $\varepsilon = \sqrt{2}/6$ covering $G^\star_1$**). *We can compute the minimum covers of* $\mathbf{C}(\sqrt{2}/6, G^\star_1, 1)$*, using the formula:*

$$\lceil \mathcal{H}^{dim_H(G^\star_1)}(G^\star_1)/(\sqrt{2}/6)\rceil$$

where $\lceil \mathcal{H}^{\dim_H(G^\star_1)}(G^\star_1)/(\sqrt{2}/6)\rceil = \lceil \text{Length}([-1,1])/(\sqrt{2}/6)\rceil = \lceil 2/(\sqrt{2}/6)\rceil = \lceil 6\sqrt{2}\rceil = \lceil 6(1.4)\rceil = \lceil 8+.4\rceil = 9$). Note there are other examples of $\mathbf{C}(\sqrt{2}/6, G^\star_1, \omega)$ for different ω. Here is another case:

FIGURE 4. This is similar to figure 3, except the start-points of the covers are shifted all the way to the left.



which can be defined (see Equation 187 for comparison):

$$\left\{\left\{(x,f(x)):\frac{6-9\sqrt{2}}{6}\le x<\frac{6-8\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-8\sqrt{2}}{6}\le x<\frac{6-7\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-7\sqrt{2}}{6}\le x<\frac{6-6\sqrt{2}}{6}\right\}\right.$$
$$\left\{(x,f(x)):\frac{6-6\sqrt{2}}{6}\le x<\frac{6-5\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-5\sqrt{2}}{6}\le x<\frac{6-4\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-4\sqrt{2}}{6}\le x<\frac{6-3\sqrt{2}}{6}\right\} \quad (189)$$
$$\left.\left\{(x,f(x)):\frac{6-3\sqrt{2}}{6}\le x<\frac{6-2\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-2\sqrt{2}}{6}\le x<\frac{6-\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-\sqrt{2}}{6}\le x\le 1\right\}\right\}$$

In the case of $G_1^\star$, there are uncountable *different covers* $\mathbf{C}(\sqrt{2}/6,G_1^\star,\omega)$ which can be used. For instance, when $0\le\alpha\le(12-9\sqrt{2})/6$ (i.e., $\omega=\alpha+1$) consider:

$$\left\{\left\{(x,f(x)):\alpha-1+\alpha\le x<\alpha+\frac{\sqrt{2}-6}{6}\right\},\left\{(x,f(x)):\alpha+\frac{\sqrt{2}-6}{6}\le x<\alpha+\frac{2\sqrt{2}-6}{6}\right\},\left\{(x,f(x)):\alpha+\frac{2\sqrt{2}-6}{6}\le x<\alpha+\frac{3\sqrt{2}-6}{6}\right\}\right.$$
$$\left\{(x,f(x)):\alpha+\frac{3\sqrt{2}-6}{6}\le x<\alpha+\frac{4\sqrt{2}-6}{6}\right\},\left\{(x,f(x)):\alpha+\frac{4\sqrt{2}-6}{6}\le x<\alpha+\frac{5\sqrt{2}-6}{6}\right\},$$
$$\left\{(x,f(x)):\alpha+\frac{5\sqrt{2}-6}{6}\le x<\alpha+\frac{6\sqrt{2}-6}{6}\right\},\left\{(x,f(x)):\alpha+\frac{6\sqrt{2}-6}{6}\le x<\alpha+\frac{7\sqrt{2}-6}{6}\right\} \quad (190)$$
$$\left.\left\{(x,f(x)):\alpha+\frac{7\sqrt{2}-6}{6}\le x<\alpha+\frac{8\sqrt{2}-6}{6}\right\},\left\{(x,f(x)):\alpha+\frac{8\sqrt{2}-6}{6}\le x\le\alpha+\frac{9\sqrt{2}-6}{6}\right\}\right\}$$

When $\alpha=0$ and $\omega=1$, we get figure 3 and when $\alpha=(12-9\sqrt{2})/6$ and $\omega=(18-9\sqrt{2})/6$, we get figure 4

8.2. **Example of Section 5.3.1, step 2.** . Suppose:

(1) $A=\mathbb{R}$
(2) When defining $f:A\to\mathbb{R}$: i.e.,

$$f(x)=\begin{cases}1 & x<0\\ -1 & 0\le x<0.5\\ 0.5 & 0.5\le x\end{cases} \quad (191)$$

(3) $\{G_r^\star:r\in\mathcal{A}(A):=\mathbb{R}^+\}=\{\{(x,f_r^\star(x)):-r\le x\le r\}:r\in\mathcal{A}(A):=\mathbb{R}^+\}=\{\{(x,f(x)):-r\le x\le r\}:r\in\mathcal{A}(A):=\mathbb{R}^+\}$
(4) $G_1^\star=\{(x,f(x)):-1\le x\le 1\}$
(5) $\mathbf{C}(\sqrt{2}/6,G_1^\star,1)$, using Equation 188 and Figure 3, which is *approximately*

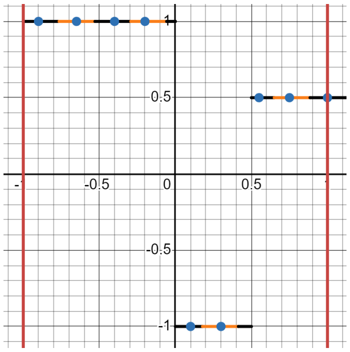
$$\{\,\{(x,f(x)):-1\le x<-.764\},\{(x,f(x)):-.764\le x<-.528\},\{(x,f(x)):-.528\le x<-.293\}$$
$$\{(x,f(x)):-.293\le x<-.057\},\{(x,f(x)):-.057\le x<.178\},\{(x,f(x)):.178\le x<.414\} \quad (192)$$
$$\{(x,f(x)):.414\le x<.65\},\{(x,f(x)):.65\le x<.886\},\{(x,f(x)):.886\le x\le 1.121\}\,\}$$

Then, an example of $\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)$ is:

$$\{(-.9,1),(-.65,1),(-.4,1),(-.2,1),(.1,-1),(.3,-1),(.55,.5),(.75,.5),(1,.5)\} \quad (193)$$

Below, we illustrate the sample: i.e., the set of all blue points *in each orange and black line of* $\mathbf{C}(\sqrt{2}/6,G_1^\star,1)$ covering $G_1^\star$:

FIGURE 5. The blue points are the "sample points", the alternative black and orange lines are the "covers", and the red lines are the *smallest* & *largest* x-coordinates of $G_1^\star$.



Note, there are multiple samples that can be taken, as long as one sample point is taken from each cover in $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$.

8.3. Example of Section 5.3.1, step 3.

Suppose

1. $A = \mathbb{R}$
2. When defining $f : A \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \le x < 0.5 \\ 0.5 & 0.5 \le x \end{cases} \tag{194}$$

3. $\{G_r^\star : r \in \mathcal{A}(A) := \mathbb{R}^+\} = \{\{(x, f_r^\star(x)) : -r \le x \le r\} : r \in \mathcal{A}(A) := \mathbb{R}^+\} = \{\{(x, f(x)) : -r \le x \le r\} : r \in \mathcal{A}(A) := \mathbb{R}^+\}$
4. $G_1^\star = \{(x, f(x)) : -1 \le x \le 1\}$
5. $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$, using Equation 188 and Figure 3, is approx.

$$\begin{aligned} \{\, &\{(x, f(x)) : -1 \le x < -.764\}, \{(x, f(x)) : -.764 \le x < -.528\}, \{(x, f(x)) : -.528 \le x < -.293\} \\ &\{(x, f(x)) : -.293 \le x < -.057\}, \{(x, f(x)) : -.057 \le x < .178\}, \{(x, f(x)) : .178 \le x < .414\} \\ &\{(x, f(x)) : .414 \le x < .65\}, \{(x, f(x)) : .65 \le x < .886\}, \{(x, f(x)) : .886 \le x \le 1.121\} \,\} \end{aligned} \tag{195}$$

6. $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$, using Equation 193, is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{196}$$

Therefore, consider the following process:

8.3.1. *Step 3a.* If $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{197}$$

suppose $\mathbf{x}_0 = (-.9, 1)$. Note, the following:

1. $\mathbf{x}_1 = (-.65, 1)$ is the next point in the "pathway" since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ with the smallest 2-d Euclidean distance to $\mathbf{x}_0$ instead of $\mathbf{x}_0$.
2. $\mathbf{x}_2 = (-.4, 1)$ is the third point since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ with the smallest 2-d Euclidean distance to $\mathbf{x}_1$ instead of $\mathbf{x}_0$ and $\mathbf{x}_1$.
3. $\mathbf{x}_3 = (-.2, 1)$ is the fourth point since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ with the smallest 2-d Euclidean distance to $\mathbf{x}_2$ instead of $\mathbf{x}_0$, $\mathbf{x}_1$, and $\mathbf{x}_2$.
4. we continue this process, where the "**pathway**" of $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ is:

$$(-.9, 1) \to (-.65, 1) \to (-.4, 1) \to (-.2, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1) \tag{198}$$

Note 12. *If more than one point has the minimum 2-d Euclidean distance from* $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, *etc. take all potential pathways: e.g., using the sample in Equation 197, if* $\mathbf{x}_0 = (-.65, 1)$, *then since* $(-.9, 1)$ *and* $(-.4, 1)$ *have the smallest Euclidean distance to* $(-.65, 1)$, *take* ***two*** *pathways:*

$$(-.65, 1) \to (-.9, 1) \to (-.4, 1) \to (-.2, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1)$$

and also:

$$(-.65,1)\to(-.4,1)\to(-.2,1)\to(-.9,1)\to(.55,.5)\to(.75,.5)\to(1,.5)\to(.3,-1)\to(.1,-1)$$

8.3.2. *Step 3b.* Next, take the length of all line segments in each pathway. In other words, suppose $d(P,Q)$ is the n-th dim. Euclidean distance between points $P,Q\in\mathbb{R}^n$. Using the pathway in Equation 198, we want:

$$\begin{aligned}&\{d((-.9,1),(-.65,1)),d((-.65,1),(-.4,1)),d((-.4,1),(-.2,1)),d((-.2,1),(.55,.5)),\\&d((.55,.5),(.75,.5)),d((.75,.5),(1,.5)),d((1,.5),(.3,-1)),d((.3,-1),(.1,-1))\}\end{aligned}\tag{199}$$

Whose distances can be approximated as:

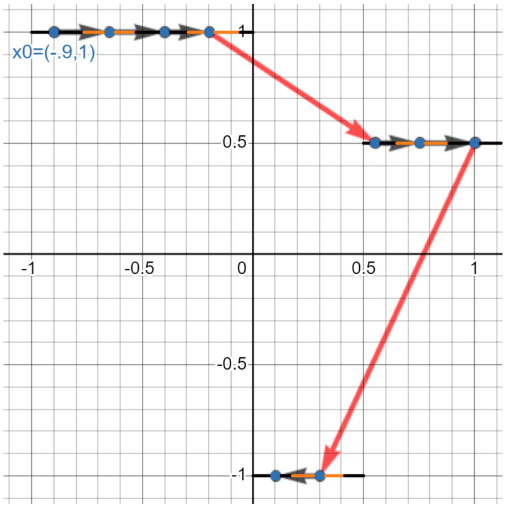
$$\{.25,.25,.2,.901389,.2,.25,1.655295,.2\}$$

Also, we see the outliers [9] are .901389 and 1.655295 (i.e., notice that the outliers are more prominent for $\varepsilon\ll\sqrt{2}/6$). Therefore, remove .901389 and 1.655295 from our set of lengths:

$$\{.25,.25,.2,.2,.25,.2\}$$

This is illustrated using:

FIGURE 6. The black arrows are the "pathways" whose lengths aren't outliers. The length of the red arrows in the pathway are outliers.



Hence, when $\mathbf{x}_0=(-.9,1)$, using Section 5.3.1 step 3b & Equation 197, we note:

$$\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1))=\{.25,.25,.2,.2,.25,.2\}\tag{200}$$

8.3.3. *Step 3c.* To convert the set of distances in Equation 200 into a probability distribution, we take:

$$\sum_{x\in\{.25,.25,.2,.2,.25,.2\}}x=.25+.25+.2+.2+.25+.2=1.35\tag{201}$$

Then divide each element in $\{.25,.25,.2,.2,.25,.2\}$ by 1.35

$$\{.25/(1.35),.25/(1.35),.2/(1.35),.2/(1.35),.25/(1.35),.2/(1.35)\}$$

which gives us the probability distribution:

$$\{5/27,5/27,4/27,4/27,5/27,4/27\}$$

Hence,

$$\mathbb{P}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))=\{5/27,5/27,4/27,4/27,5/27,4/27\}\tag{202}$$

8.3.4. *Step 3d.* Take the shannon entropy of Equation 202:

$$
\begin{aligned}
&\mathrm{E}(\mathbb{P}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))) = \\
&\sum_{x\in\mathbb{P}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))} -x\log_2 x = \sum_{x\in\{5/27,5/27,4/27,4/27,5/27,4/27\}} -x\log_2 x = \\
&-(5/27)\log_2(5/27)-(5/27)\log_2(5/27)-(4/27)\log_2(4/27)-(4/27)\log_2(4/27)-(5/27)\log_2(5/27)-(4/27)\log_2(5/27) = \\
&-(15/27)\log_2(5/27)-(12/27)\log_2(4/27)\approx 2.57604
\end{aligned}
$$

We shorten $\mathrm{E}(\mathbb{P}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1))))$ to $\mathrm{E}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))$, giving us:

$$\mathrm{E}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.57604 \tag{203}$$

8.3.5. *Step 3e.* Take the entropy, w.r.t. all pathways, of the sample:

$$\{(-.9,1),(-.65,1),(-.4,1),(-.2,1),(.1,-1),(.3,-1),(.55,.5),(.75,.5),(1,.5)\} \tag{204}$$

In other words, we'll compute:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1))) = \sup_{\mathbf{x}_0\in\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)} \mathrm{E}(\mathcal{L}(\mathbf{x}_0,\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))$$

We do this by repeating Section 8.3.1-Section 8.3.4 for different $\mathbf{x}_0\in\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))$ (i.e., in the equation with multiple values, see note 12)

$$
\begin{aligned}
\mathrm{E}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))&\approx 2.57604 \tag{205}\\
\mathrm{E}(\mathcal{L}((-.65,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))&\approx 2.3131, 2.377604 \tag{206}\\
\mathrm{E}(\mathcal{L}((-.4,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))&\approx 2.3131 \tag{207}\\
\mathrm{E}(\mathcal{L}((-.2,-1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))&\approx 2.57604 \tag{208}\\
\mathrm{E}(\mathcal{L}((-.1,-1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))&\approx 1.86094 \tag{209}\\
\mathrm{E}(\mathcal{L}((-.3,-1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))&\approx 1.85289 \tag{210}\\
\mathrm{E}(\mathcal{L}((.55,.5),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))&\approx 2.08327 \tag{211}\\
\mathrm{E}(\mathcal{L}((.75,.5),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))&\approx 2.31185 \tag{212}\\
\mathrm{E}(\mathcal{L}((1,.5),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))&\approx 2.2622 \tag{213}
\end{aligned}
$$

Hence, since the largest value out of Equation 205-213 is 2.57604:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1))) = \sup_{\mathbf{x}_0\in\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)} \mathrm{E}(\mathcal{L}(\mathbf{x}_0,\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))\approx 2.57604$$

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