DEFINING A MEASURE OF DISCONTINUITY (V3)

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Abstract. Let $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$, where $f : X \to Y$ is a function. We want to find a measure of discontinuity of f between zero and positive infinity, where the more "disconnected" the graph of f, the larger the measure. Note, the measure continuity is slightly different, where other properties such as the derivative and integral must exist. The measure of the discontinuity of f is defined w.r.t. an arbitrary set $X_1 \subseteq \mathbb{R}$. For example, when f is continuous, its restriction on X_1 is continuous (i.e., X_1 is dense in X) and the restriction should have zero measure of discontinuity. In addition, when the extension of f is defined on X_1, and the Hausdorff dimension of X_1 is greater than the dimension of X, the measure of discontinuity should be positive infinity. We define the measure by "averaging" the number of times, minus one, an arbitrary vertical line intersects the topological closure of the graph f on $X \cap X_1$ with respect to the measure of continuity of f on $X \cap X_1$.

1. Motivation

Let $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ be arbitrary sets, where $f : X \to Y$ is a function. We want a measure of discontinuity of f w.r.t. an arbitrary set $X_1 \subseteq \mathbb{R}$, where the more "disconnected" the graph of f on $(X \cap X_1) \times Y$, the larger the measure. Specifically, the measure should range[1] from zero to positive infinity. In particular, the measure should be zero when f is continuous on $X \cap X_1$ and positive infinity when the graph of f is "completely disconnected" on $(X \cap X_1) \times Y$.[2] Notice, this is measured with the d-dimensional Hausdorff measure on the Borel σ-algebra, which we denote $\mathcal{H}^d(\cdot)$ such that $d \in [0, 1]$ and $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension.

To understand discontinuity, we need to understand continuity. Informally, f is continuous, when all points in the subset of the graph of f—which cannot be approximated "infinitely close" using points on a positive measure subset of the graph—have zero measure. This means a discrete function, with a non-empty domain, is non-continuous. However, a function whose graph has a closure[3], which is a function defined on the smallest interval containing the original domain, is continuous. (Note, in Section 2.1, we define a rigorous definition of continuity.)

In topology, when the function f is continuous, its restriction to a set dense in X is continuous and has to have zero measure of discontinuity: the domain $X' = X \cap X_1$ of the restriction $f|_{X'}$ has a Hausdorff dimension less than or equal to the dimension of X. Thus, when $\dim_{\mathrm{H}}(X_1 \cap X) < \dim_{\mathrm{H}}(X)$, the measure of discontinuity should be zero. In contrast, when $\dim_{\mathrm{H}}(X_1) > \dim_{\mathrm{H}}(X)$, the measure should be positive infinity: e.g., the function f is not continuous on the domain of its extension $F : X_1 \supset X \to Y$, since $X_1 \setminus X$ has a positive Hausdorff measure in the dimension of X_1. Thus, we are left with $\dim_{\mathrm{H}}(X \cap X_1) = \dim_{\mathrm{H}}(X)$, where the measure can be any number in $[0, +\infty]$ based on intuitive reasoning. (In Section 2.2, we explain this paragraph in more detail.)

In Section 3, we attempt to define a measure which satisfies the previous paragraph and the criteria in Section 2.2. The measure is quite long and might not be useful for "simple" functions; however, it might have use for *indirect* and *sophisticated* functions. Consider, for example:

> Let $f : \mathbb{R} \to \mathbb{R}$ and $q : \{0, 1, 2, 3\} \to \{0, 1, 2, 3\}$, where $q = \{(0, 1), (1, 3), (2, 0), (3, 2)\}$.
> Suppose $x \in \mathbb{R}$, where the base 4 expansion of $x = a_1 a_2 \ldots a_m . a_{m+1} a_{m+2} \ldots$ such that a_1 is the left most non-zero digit *and* a_{m+j}, for some $j \in \mathbb{N} \cup \{+\infty\}$, is the last non-zero digit before a trail of zeros *or else* $j = +\infty$.
> Hence, for all $i \in \mathbb{N}$, $c_i = \mathrm{mod}(\sum_{p=1}^{i} \mathrm{mod}(p \cdot q^p(a_i), 4), 4)$ and $f(x) = c_1 c_2 \ldots c_m . c_{m+1} c_{m+2} \ldots$.

Date: August 1, 2025.

[1] The measure in Section 3.1, page 4 is rational when $X \cap X_1$ is bounded and real when $X \cap X_1$ is unbounded

[2] See Section 4.4.1, page 33, where $(X \cap X_1) \times Y = \mathbb{R} \times \mathbb{R}$

[3] topological closure

A simpler *sophisticated* example is the following:

Suppose $f : \mathbb{Q} \to \mathbb{R}$. When $x = p/q$ for all coprime integers $p, q \in \mathbb{Z}$, then $f(p/q) = (p+q)/(pq)$

However, *direct* and *applicable* versions can be found in Section 4.6 and Section 4.7 (p. 33).

In Section 3.3, we provide explicit examples which hint that we have some idea behind the construction of our measure in Section 3.1. This does not fully prove that we found a measure which satisfies all the criteria, but the measure in Section 3.1 can be used to find a simpler version.

2. Preliminaries

Note, the definition of continuity is:

2.1. **Definition of Continuity.** Suppose $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ are arbitrary sets. The continuity of $f : X \to Y$ at $x_0 \in X$ means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X \setminus \{x_0\}$

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \epsilon$$

Hence, f is continuous in arbitrary set $X_1 \subseteq \mathbb{R}$, where:

(1) $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension
(2) $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimensions on the Borel σ-algebra

whenever:

$$C_{\mathcal{M}}(f, X_1) = \mathcal{H}^{\dim_{\mathrm{H}}(X_1)}(X_1 \setminus X) = 0$$

2.1.1. *Example 1 of Definition.* When $X = \mathbb{Q}$, $Y = \mathbb{R}$ and $f(x) = x$ for all $x \in \mathbb{Q}$, f is continuous on $X_1 = \mathbb{Q}$ but discontinuous on $X_1 = \mathbb{R}$.

2.1.2. *Example 2 of Definition.* When $X = \mathbb{N}$, $Y = \mathbb{R}$ and $f(x) = x$ for all $x \in \mathbb{N}$, f is not continuous on $X_1 = \mathbb{N}$, $X_1 = \mathbb{Q}$, and $X_1 = \mathbb{R}$.

2.1.3. *Example 3 of Definition.* When $X = \mathbb{R}$, $Y = \mathbb{R}$ and $f(x) = x$ for all $x \in \mathbb{R}$, f is continuous on any dense subset X_1 of $\mathbb{R}$

2.2. **Criteria For Measure.**

2.2.1. *Explanation.* Suppose $U \subseteq X$ is an arbitrary set, and $f|_U$ is a restriction of the function $f : X \to Y$. Let $F : V \to Y$ is an extension of the function f, where $X \subseteq V$ is an arbitrary set and $F|_X = f$

In topology, when the function $f : X \to Y$ is continuous, its restriction is continuous and should have zero measure of discontinuity. (This could be proven with Section 2.1, where $C_{\mathcal{M}}(f, U) = 0$.) Hence, since the domain U of the restriction $f|_U$ has a $\dim_{\mathrm{H}}(U)$ less than or equal to $\dim_{\mathrm{H}}(X)$, the measure of the discontinuity of $f|_U$ should be zero. In contrast, when $V \supseteq X$ is an arbitrary set and $\dim_{\mathrm{H}}(V) > \dim_{\mathrm{H}}(X)$, the measure should be positive infinity: e.g., the function f is not continuous in its extension $F : V \to Y$ since $C_{\mathcal{M}}(f, V) > 0$. Thus, we are left with either $\dim_{\mathrm{H}}(U) = \dim_{\mathrm{H}}(X)$ or $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{H}}(V)$, where the measure can be any number in $[0, +\infty]$.

Note, when $\dim_{\mathrm{H}}(U) = \dim_{\mathrm{H}}(X)$ or $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{H}}(V)$, we use intuition to justify how a measure of discontinuity should be defined. (Intuition is frowned upon; however, some intuition has to be used.) In the case of this paper, when f is continuous, the closure[4] of the graph of f is one function continuous on $\mathbb{R}$.[5] Hence, an arbitrary vertical line intersects with the closure once and the measure of discontinuity should be zero. Similarly, the closure of the graph of the Dirichlet function is two functions continuous on $\mathbb{R}$,[5] where an arbitrary vertical line intersects the closure twice. (In Sections 4.6 and 4.7, on page 33, we show that not all vertical lines intersecting the domain of f have to intersect the closure the same number of times.) Hence, we "average" the amount of times *minus one* that the vertical lines intersect the closure of the graph, w.r.t. the measure of continuity in Section 2.1. (The intuition is further explained below.)

[4] topological closure
[5] See Section 4.1, page 33, where $X_1 = \mathbb{R}$

2.2.2. *Criteria.* Suppose $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ are arbitrary sets, where $f : X \to Y$ is a function. Note, $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension, $X_1 \subseteq \mathbb{R}$ is an arbitrary set, and $X' = X \cap X_1$.

The measure of discontinuity of f on X_1 should range from zero to positive infinity, where:

(1) The more "disconnected" the graph of f on $X' \times Y$, the higher the measure of discontinuity.
(2) When $X \cap X_1$ is empty, the measure is zero, regardless of $\dim_{\mathrm{H}}(X_1)$ and $\dim_{\mathrm{H}}(X \cap X_1)$
(3) When the closure of the graph of $f|_{X'}$ is empty but $X \cap X_1$ is non-empty, the measure is $+\infty$ regardless of $\dim_{\mathrm{H}}(X_1)$ and $\dim_{\mathrm{H}}(X \cap X_1)$
(4) When the function is continuous on X_1,[6] the measure is zero.
(5) When $f|_{X'}$ is "hyper-discontinuous" (Section 4.3, page 33), the measure is positive infinity.
(6) When the closure of the graph of f can be split into a minimum of n functions continuous on a positive measure subset $\mathbb{X} \subseteq X_1$ (Section 4.2, page 33) such that every vertical line, where their x-intercept is an element of $X \cap X_1$, intersects the closure m to n times ($m < n$):
   (a) If $\dim_{\mathrm{H}}(X_1 \cap X) < \dim_{\mathrm{H}}(X)$, the measure is a number between $m-1$ and $n-1$, and corresponds to the weighted average $\mathcal{D}(f)$ (Equation 1) where:
      - the variable $\mathfrak{c}$ is the number of the times the vertical line intersects the closure with respect to its x-intercept
      - $\limsup_{j\to\infty} \mathbf{X}'_j = \liminf_{j\to\infty} \mathbf{X}'_j = X'$ (i.e., the set theoretic limit[7]) such that $0 < \mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}'_j)$ $< +\infty$ for all $j \in \mathbb{N}$
      - the arbitrary set $\overline{X_{\mathfrak{c}}} \subseteq X \cap X_1$ has the largest Hausdorff measure in the Hausdorff dimension of $X \cap X_1$, such that the vertical line for all $x \in \overline{X_{\mathfrak{c}}}$ intersects the closure $\mathfrak{c}$ times ($m \leq \mathfrak{c} \leq n$)

$$\mathcal{D}(f) = \lim_{(a,b)\to(-\infty,\infty)} \left(\liminf_{j\to\infty} \frac{\sum_{\mathfrak{c}=m}^{n} (\mathfrak{c}-1)\cdot \mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}'_j \cap \overline{X_{\mathfrak{c}}} \cap (a,b))}{\mathcal{H}^{\dim_{\mathrm{H}}(X')}((X \cap \mathbf{X}'_j \cap X_1) \cap (a,b))} \right) \tag{1}$$

$$= \lim_{(a,b)\to(-\infty,\infty)} \left(\limsup_{j\to\infty} \frac{\sum_{\mathfrak{c}=m}^{n} (\mathfrak{c}-1)\cdot \mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}'_j \cap \overline{X_{\mathfrak{c}}} \cap (a,b))}{\mathcal{H}^{\dim_{\mathrm{H}}(X')}((X \cap \mathbf{X}'_j \cap X_1) \cap (a,b))} \right) \tag{2}$$

   (b) If $\dim_{\mathrm{H}}(X_1 \cap X) = \dim_{\mathrm{H}}(X)$, the measure is a number between $m-1$ and $n-1$, and corresponds to the weighted average $\mathcal{D}(f)$.
   (c) If $\dim_{\mathrm{H}}(X_1) > \dim_{\mathrm{H}}(X)$, the measure is positive infinity
(7) When the graph of f is dense in the closure of $X \times Y$, where we remove the subset of the graph of f with zero Hausdorff measure in its dimension and are left with the minimum of n function continuous on X_1:[6]
   (a) If $\dim_{\mathrm{H}}(X_1 \cap X) < \dim_{\mathrm{H}}(X)$, the measure is less than or equal to $n-1$
   (b) If $\dim_{\mathrm{H}}(X_1 \cap X) = \dim_{\mathrm{H}}(X)$, the measure is $n-1$
   (c) If $\dim_{\mathrm{H}}(X_1) > \dim_{\mathrm{H}}(X)$, the measure is positive infinity
(8) When f is everywhere surjective (Section 4.4, page 33), where its graph has zero Hausdorff measure in its dimension, the measure is $+\infty$.

2.2.3. *Question.* Is there a measure of discontinuity that gives what I want? What applications can this measure have?

3. Possible Answer To Section 2.2.2

In Section 3.1, we define a measure that satisfies the criteria in Section 2.2.2; however, we need direct evidence. We begin by explaining our motivations for the components $z(\varepsilon, X'(a,b))$, $r(\varepsilon, X'(a,b))$, $h(\varepsilon, X'(a,b))$, and $\mathtt{C}_0(d_1, \ell, X'(a,b))$ defined within the measure of discontinuity (eq. 3-4). In Section 3.2, we use examples to explain why we chose the measure. The evidence is presented in Section 3.3.

[6] See Section 4.1, page 33, where X_1 is an arbitrary set
[7] See Section 4.5, page 33

3.1. **Measure.** Suppose,

- $X \subseteq \mathbb{R}$, $X_1 \subseteq \mathbb{R}$, and $Y \subseteq \mathbb{R}$ are arbitrary sets
- $f : X \to Y$ is a function
- $d \in [0, 1]$
- $\mathcal{H}^d(\cdot)$ is the d-dimensional Hausdorff measure on the Borel σ-algebra
- $\dim_{\mathrm{H}}(\cdot)$ or $d'(\cdot)$ is the Hausdorff dimension
- $d_1 = \dim_{\mathrm{H}}(X_1)$
- $X' = X \cap X_1$
- $X' \cap (a, b) = X'(a, b)$
- $z(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & \mathcal{H}^{d_1}(X'(a,b)) = 0,\, \varepsilon > 0 \\ 1/\varepsilon & 0 < \mathcal{H}^{d_1}(X'(a,b)) \le +\infty,\, \varepsilon > 0 \end{cases}$

  Motivation of $z(\varepsilon, X'(a,b))$.

  The motive behind $z(\varepsilon, X'(a,b))$ is to equal ε, when either $X'(a,b)$ is the empty set or $\dim_{\mathrm{H}}(X_1) > \dim_{\mathrm{H}}(X)$. In addition, $z(\varepsilon, X'(a,b))$ is equivalent to $1/\varepsilon$, when $X'(a,b)$ is non-empty and $\dim_{\mathrm{H}}(X \cap X_1) \le \dim_{\mathrm{H}}(X)$. Note, $\lim_{\varepsilon\to 0} \varepsilon = 0$ and $\lim_{\varepsilon \to 0} 1/\varepsilon = +\infty$. (We do not compute either limits until Equation 3 page 5).
- $\limsup_{j\to\infty} X_j(a,b) = \liminf_{j\to\infty} X_j(a,b) = X'(a,b)$ (i.e., the set-theoretic limit[8]) such that $0 < \mathcal{H}^{d_1}(X_j(a,b)) < +\infty$ for all $j \in \mathbb{N}$
- $\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^{d_1}(X_j(a,b))\}$

  Motivation of $\mathbf{R}(X_j(a,b))$.

  The motive behind $\mathbf{R}(X_j(a,b))$ is to equal a finite number, when $\dim_{\mathrm{H}}(X \cap X_1) \le \dim_{\mathrm{H}}(X)$, such that the closure of the graph of f can be split into c functions continuous on a positive measure subset $\mathbb{X} \subseteq X_1$[9]. Otherwise, we want $\mathbf{R}(X_j(a,b)) = \inf(\emptyset) = +\infty$ is necessary for $\dim_{\mathrm{H}}(X_1) > \dim_{\mathrm{H}}(X)$.
- $|\cdot|$ is the absolute value function
- $r(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & 0 \le \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| < +\infty,\, \varepsilon > 0 \\ 1/\varepsilon & \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = +\infty,\, \varepsilon > 0 \end{cases}$

  Motivation of $r(\varepsilon, X'(a,b))$.

  The motive behind $r(\varepsilon, X'(a,b))$ is to equal $1/\varepsilon$, when either $\mathbf{R}(X_j(a,b)) = +\infty$, the function $f : \mathbb{Q} \cap [0,1] \to \mathbb{R}$ is hyper-discontinuous[10] or the map $f : X' \to Y$ is everywhere surjective[11], where its graph has zero Hausdorff measure in its dimension. (Note, $\lim_{\varepsilon\to 0} 1/\varepsilon = +\infty$, which we do not compute until Equation 3 page 5.) For instance, consider Section 4.3.1 and Section 4.4.1 on page 33.
- $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$
- $\limsup_{j\to\infty} \mathsf{G}_j(X'(a,b)) = \liminf_{j\to\infty} \mathsf{G}_j(X'(a,b)) = G(X'(a,b))$ (i.e., the set theoretic limit[8]) such that $0 < \mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) < +\infty$ for all $j \in \mathbb{N}$
- $h(\varepsilon, X'(a,b)) = \begin{cases} 1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = 0,\, \varepsilon > 0 \\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0, +\infty)), \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \ne 0 \end{cases}$

  Motivation of $h(\varepsilon, X'(a,b))$.

  The motive behind $h(\varepsilon, X'(a,b))$ is to equal $1/\varepsilon$, when X is empty or the function $f : X' \to Y$ is everywhere surjective[11], such that its graph has zero Hausdorff measure in its dimension. Note, $\lim_{\varepsilon\to 0} 1/\varepsilon = +\infty$, which we do not compute until Equation 3 page 5.
- $\mathcal{P}_k(d_1, G(X'(a,b))) = \left\{\mathbf{G} \subseteq G(X'(a,b)) : \mathcal{H}^{d_1}(\mathbf{G}) = k\right\}$
- $G_k(d_1, X'(a,b)) \in \mathcal{P}_k(d_1, G(X'(a,b)))$

[8] See Section 4.5, page 33
[9] See Section 4.2, page 33
[10] See Section 4.3, page 33
[11] See Section 4.4, page 33

- $\mathbb{G}_0(d_1, X'(a,b))$ is the set of all limit points of $G(X'(a,b)) \setminus G_0(d_1, X'(a,b))$
- $\#|\cdot|$ is the counting measure
- $\mathbf{C}(X) = \begin{cases} 1/\varepsilon & \#|X| = 0, \varepsilon > 0 \\ \#|X| & 0 < \#|X| < +\infty \\ \varepsilon & \#|X| = +\infty,\ \varepsilon > 0 \end{cases}$

  Motivation of $\mathbf{C}(X)$.

  The motive behind $\mathbf{C}(X)$ is to equal:
  - $1/\varepsilon$, when X is empty
  - $1/\varepsilon$, when $\dim_{\mathrm{H}}(X_1) > \dim_{\mathrm{H}}(X)$
  - A finite value, when $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{H}}(X \cap X_1)$, such that the closure of graph(f) can be split into finitely many functions continuous on a positive measure subset $\mathbb{X} \subseteq X_1$[12]
  - Positive infinity, when the map $f : X' \to Y$ is everywhere surjective[13], where its graph has zero Hausdorff measure in its dimension.
- $\ell(X'(a,b)) \subset \mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$
- $\mathtt{C}_0(d_1, \ell, X'(a,b)) = \inf_{G_0(d_1,X'(a,b)) \in \mathcal{P}_0(d_1,X'(a,b))} \mathbf{C}(\mathbb{G}_0(d_1, X'(a,b)) \cap \ell(X'(a,b)))$
- When defining:
  - $\limsup_{j\to\infty} \mathbf{X}_j(a,b) = \liminf_{j\to\infty} \mathbf{X}_j(a,b) = X'(a,b)$ (i.e., the set theoretic limit[8]) such that $0 < \mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b)) < +\infty$ for all $j \in \mathbb{N}$
  - $\mathbf{M}(\varepsilon, d_1, \ell, X'(a,b)) = z(\varepsilon, X'(a,b)) \cdot r(\varepsilon, X'(a,b)) \cdot \mathtt{C}_0(d_1, \ell, X'(a,b)) \cdot h(\varepsilon, X'(a,b))$

    Motivation of $\mathbf{M}(\varepsilon, d_1, \ell, X'(a,b))$.

    See Section 3.2 and Section 3.3 for examples and explanations. Note, $\lim_{\varepsilon\to 0}$ $\mathbf{M}(\varepsilon, d_1, \ell, X'(a,b))$ is the same as the column $\lim_{\varepsilon\to 0}(z \cdot r \cdot h \cdot \mathtt{C}_0)$ in Table 1.

$$\mathcal{M}_j(a,b,d_1,t) = \left(\frac{\sum\limits_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}} \max\{0, \min\{\mathfrak{c}-1, t\}\} \cdot \sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\left(\left\{B \cap \mathbf{X}_j(a,b) : B \subseteq X'(a,b),\ \lim\limits_{\varepsilon\to 0} \mathbf{M}(\varepsilon, d_1, \ell, B) = \mathfrak{c}\right\}\right)\right)}{\max\{1/t, \mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b))\}} \right) \tag{3}$$

$\mathcal{D}_{d_1}$ is the d_1-dimensional measure of discontinuity (when the limit exists)

$$\mathcal{D}_{d_1}(f) = \lim_{(a,b)\to(-\infty,\infty)} \left(\lim_{t\to\infty} \left(\liminf_{j\to\infty} \mathcal{M}_j(a,b,d_1,t) \right) \right) \tag{4}$$

$$= \lim_{(a,b)\to(-\infty,\infty)} \left(\lim_{t\to\infty} \left(\limsup_{j\to\infty} \mathcal{M}_j(a,b,d_1,t) \right) \right) \tag{5}$$

3.1.1. *Note.* (If there exists $j \in \mathbb{N}$, where $\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b)) = +\infty$, replace $\mathcal{H}^{\dim_{\mathrm{H}}(X')}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33].)

3.2. **Explanation of Measure.** In Section 3.2, we give examples of f that satisfy the criteria in Section 2.2.2, applying the measure of discontinuity to all examples in Section 3.3.

3.2.1. *Summary of Cases.* In Table 1, we visualize the computation of the measure of discontinuity $\mathcal{D}_{d_1}(f)$ for all the cases of f in Section 3.3. The table contains the output of each component of $\mathcal{M}_j(a,b,d_1,t)$ (Equation 3, pg. 5) which is used to compute $\mathcal{D}_{d_1}(f)$ in the last column.

Table 1: Visualization for Computing The Measure of Discontinuity. See Section 3.1. Note, $X' = X \cap X_1$ and $X'(a,b) = (X \cap X_1) \cap (a,b)$

[12] See Section 4.2, page 33

[13] See Section 4.4, page 33

$f: X \to Y$	d_1	$\dim_{\mathrm{H}}(X)$	$z(\varepsilon, X'(a,b))$	$r(\varepsilon, X'(a,b))$	$h(\varepsilon, X'(a,b))$	$\mathsf{c}_0(d_1, \ell, X'(a,b))$	$\lim_{\varepsilon\to 0}(z \cdot r \cdot h \cdot \mathsf{c}_0)$	$\mathcal{D}_{d_1}(f)$
Case 1 (p. 6)	0	0	ε	ε	$1/\varepsilon$	$1/\varepsilon$	1	0
Case 2 (p. 8)	1	0	ε	ε	$1/\varepsilon$	$1/\varepsilon$	1	0
Case 3 (p. 8)	0	0	$1/\varepsilon$	ε	1	$1/\varepsilon$	$+\infty$	$+\infty$
Case 4 (p. 11)	0	1	$1/\varepsilon$	ε	1	$0 \le \mathsf{c}_0 \le \mathbf{c_1}$	c_0	$0 \le \mathcal{D}_{d_1} \le \mathbf{c_1} - 1$
Case 5 (p. 15)	1	0	ε	$1/\varepsilon$	1	$1/\varepsilon$	$+\infty$	$+\infty$
Case 6 (p. 18)	1	1	$1/\varepsilon$	ε	1	$0 \le \mathsf{c}_0 \le \mathbf{c_1}$	c_0	$0 \le \mathcal{D}_{d_1} \le \mathbf{c_1} - 1$
Case 7 (p. 21)	0	0	$1/\varepsilon$	ϵ	1	$0 \le \mathsf{c}_0 \le \mathbf{c_1}$	c_0	$0 \le \mathcal{D}_{d_1} \le \mathbf{c_1} - 1$
Case 8 (p. 22)	0	0	$1/\varepsilon$	$1/\varepsilon$	1	1	$+\infty$	$+\infty$
Case 9 (p. 25)	0	0	$1/\varepsilon$	$1/\varepsilon$	1	ε	$+\infty$	$+\infty$
Case 10 (p. 27)	1	1	$1/\varepsilon$	ε	1	1	1	0
Case 11 (p. 30)	1	1	$1/\varepsilon$	$1/\varepsilon$	$1/\varepsilon$	ε	$+\infty$	$+\infty$

If any calculations in the table are wrong, the measure is insufficient. Thus, we need to explain our reasoning in Section 3.3

3.3. Computing $\mathcal{D}_{d_1}(f)$ for each Case in Table 1.

We need to make sure the measure is correct, which is done by checking for inaccuracies in Table 1. Otherwise, the measure of discontinuity is not well defined.

3.3.1. *Case 1.* Suppose $f : X \to Y$ is a function, where $X = \emptyset$, $Y = \mathbb{R}$, $X_1 = \mathbb{Q}$, $d_1 = \dim_{\mathrm{H}}(X_1) = 0$ and $\dim_{\mathrm{H}}(X) = 0$. Hence, $X' = X \cap X_1 = \emptyset$.

> Since Case 1 is an example of Section 2.2.2, criteria 2, the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (Equation 4, page 5) should be zero.

For parts 1-5, we compute the components $z(\varepsilon, X'(a,b))$, $r(\varepsilon, X'(a,b))$, $h(\varepsilon, X'(a,b))$, $\mathsf{c}_0(d_1, \ell, X'(a,b))$ and $\lim_{\varepsilon\to 0} \mathbf{M}(\varepsilon, d_1, \ell, X'(a,b))$ (i.e., their motivation is in Section 3.1, page 4), then use these components to compute the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (part 6).

Part 1. $z(\varepsilon, X'(a,b))$

Suppose:

$$z(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & \mathcal{H}^{d_1}(X'(a,b)) = 0,\, \varepsilon > 0 \\ 1/\varepsilon & 0 < \mathcal{H}^{d_1}(X'(a,b)) \le +\infty,\, \varepsilon > 0 \end{cases}$$

since $\mathcal{H}^0(X'(a,b)) = \mathcal{H}^0(\emptyset) = 0$, therefore $z(\varepsilon, X'(a,b)) = \varepsilon$.

Part 2. $r(\varepsilon, X'(a,b))$

Suppose:

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^{d_1}(X_j(a,b))\}$$

Note, $X_j(a,b) = \{j, j+1\}$, since $\liminf_{j\to\infty} X_j(a,b) = \limsup_{j\to\infty} X_j(a,b) = \emptyset$ (i.e., the set-theoretic limit[14]) and for all $j \in \mathbb{N}$, $\mathcal{H}^0(X_j(a,b)) = 2$.

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap \{j, j+1\}) = 2\}$$

Hence, $\dim_{\mathrm{H}}(\mathrm{Range}(f)) = 0$, since the range of a function defined on the empty set is the empty set and the Hausdorff dimension of the empty set is zero (i.e., the Hausdorff dimension is non-negative). Thereby, R can be any set, where the counting measure $\#|R| \ge 2$ and $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) \ge 2$. Thus, the smallest $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R)$ can be is two, which means that $\mathbf{R}(X_j(a,b)) = 2$.

Therefore, $\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = |2-2| = 0$, for all $j \in \mathbb{N}$ and $0 \le \lim_{j\to\infty} |\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))| = 0 < +\infty$. Thus, using:

$$r(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & 0 \le \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| < +\infty,\, \varepsilon > 0 \\ 1/\varepsilon & \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = +\infty,\, \varepsilon > 0 \end{cases}$$

[14] See Section 4.5, page 33

we have $r(\varepsilon, X'(a,b)) = \varepsilon$.

Part 3. $h(\varepsilon, X'(a,b))$

If $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$, $G(X'(a,b))$ is empty, since $X'(a,b)$ is empty.
In addition, $\dim_{\mathrm{H}}(G(X'(a,b))) = d'(G(X'(a,b))) = 0$ and $\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = 0$. Hence, using:

$$h(\varepsilon, X'(a,b)) = \begin{cases} 1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = 0,\, \varepsilon > 0 \\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty)), \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0 \end{cases}$$

we have $h(\varepsilon, X'(a,b)) = 1/\varepsilon$

Part 4. $\mathtt{C}_0(d_1, \ell, X'(a,b))$

Suppose, $d_1 = 0$ and $\#|\cdot|$ is the counting measure.
Notice that $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$ and $G(X'(a,b))$ is empty since $X'(a,b)$ is empty.

Also, note that:

$$\mathcal{P}_k(d_1, G(X'(a,b))) = \left\{ \mathbf{G} \subseteq G(X'(a,b)) : \mathcal{H}^{d_1}(\mathbf{G}) = k \right\}$$

$$G_k(d_1, X'(a,b)) \in \mathcal{P}_k(d_1, G(X'(a,b)))$$

Hence, since $\mathbb{G}_0(d_1, X'(a,b))$ is the set of all limit points of $G(X'(a,b)) \setminus G_0(d_1, X'(a,b))$ where $G(X'(a,b))$ is empty, $\mathbf{G}$ is empty, and $G_0(d_1, X'(a,b))$ is empty. Therefore, $\mathbb{G}_0(d_1, X'(a,b))$ is empty.

Moreover, $\ell(X'(a,b)) \subset \mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$ and since $X'(a,b)$ is empty, $\ell(X'(a,b))$ is empty.
Hence, whenever:

$$\mathbf{C}(X) = \begin{cases} 1/\varepsilon & \#|X| = 0, \varepsilon > 0 \\ \#|X| & 0 < \#|X| < +\infty \\ \varepsilon & \#|X| = +\infty,\, \varepsilon > 0 \end{cases} \tag{6}$$

since

$$\begin{aligned} \mathtt{C}_0(d_1, \ell, X'(a,b)) = & \inf_{G_0(d_1, X'(a,b)) \in \mathcal{P}_0(d_1, X'(a,b))} \mathbf{C}(\mathbb{G}_0(d_1, X'(a,b)) \cap \ell(X'(a,b))) \\ & \inf_{G_0(d_1, X'(a,b)) \in \mathcal{P}_0(d_1, X'(a,b))} \mathbf{C}(\emptyset \cap \emptyset) = \mathbf{C}(\emptyset) \end{aligned}$$

w.r.t. the counting measure $\#|\cdot|$ and Equation 6, $\#|\emptyset| = 0$ and $\mathbf{C}(\emptyset) = 0$. Thus, $\mathtt{C}_0(d_1, \ell, X'(a,b)) = 1/\varepsilon$.

Part 5. $\lim_{\varepsilon \to 0}(z \cdot r \cdot h \cdot \mathtt{C}_0)$

In Case 1 in Section 3.3.1 Part 1, 2, 3, and 4, where:

- $z(\varepsilon, X'(a,b)) = \varepsilon$
- $r(\varepsilon, X'(a,b)) = \varepsilon$
- $h(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $\mathtt{C}_0(d_1, \ell, X'(a,b)) = 1/\varepsilon$

$$\begin{aligned} & \lim_{\varepsilon \to 0} \mathbf{M}(\varepsilon, d_1, \ell, X'(a,b)) = \\ & \lim_{\varepsilon \to 0} \left(z(\varepsilon, X'(a,b)) \cdot r(\varepsilon, X'(a,b)) \cdot h(\varepsilon, X'(a,b)) \cdot \mathtt{C}_0(d_1, \ell, X'(a,b)) \right) = \lim_{\varepsilon \to 0} \left(\varepsilon \cdot \varepsilon \cdot 1/\varepsilon \cdot 1/\varepsilon \right) = 1 \end{aligned} \tag{7}$$

Part 6. Applying The Measure of Discontinuity to f

For every $B \subseteq X'(a,b)$, $\mathfrak{c}=1$ (see Equation 7):

$$\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\lim_{\varepsilon\to 0}(\varepsilon\cdot\varepsilon\cdot 1/\varepsilon\cdot 1/\varepsilon)=1=\mathfrak{c} \tag{8}$$

Thus, when $\mathfrak{c}=0$ and $\mathfrak{c}>1$, B is the empty set. Hence, B is always the empty set, since $X'(a,b)$ is the empty set.
Therefore:

$$\mathcal{M}_j(a,b,d_1,t)= \tag{9}$$
$$\left(\frac{\sum_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\ \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b))\}}\right)=$$
$$\frac{\max\{0,\min\{0-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq\emptyset,\ \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=0\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X'(a,b))\}}+$$
$$\frac{\max\{0,\min\{1-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq\emptyset,\ \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=1\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X'(a,b))\}}+$$
$$\frac{\sum_{\mathfrak{c}\in\{2,3,\cdots\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq\emptyset,\ \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X'(a,b))\}}=$$

Once more, since B is always the empty set, the Hausdorff measure of B in its dimension is zero. Hence, Equation 9 is the same as:

$$\frac{\max\{0,\min\{-1,t\}\}\cdot\mathcal{H}^0(\emptyset)}{\max\{1/t,\mathcal{H}^0(\emptyset)\}}+\frac{\max\{0,\min\{0,t\}\}\cdot\mathcal{H}^0(\emptyset)}{\max\{1/t,\mathcal{H}^0(\emptyset)\}}+\frac{\sum_{\mathfrak{c}\in\{2,3,\cdots\}\cup\{\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^0(\emptyset)}{\max\{1/t,\mathcal{H}^0(\emptyset)\}}=$$
$$\frac{\max\{0,-1\}\cdot 0}{\max\{1/t,0\}}+\frac{\max\{0,0\}\cdot 0}{\max\{1/t,0\}}+\frac{\sum_{\mathfrak{c}\in\{2,3,\cdots\}\cup\{\infty\}}\max\{0,\mathfrak{c}-1\}\cdot 0}{\max\{1/t,0\}}=$$
$$\frac{\max\{0,-1\}\cdot 0}{1/t}+\frac{\max\{0,0\}\cdot 0}{1/t}+\frac{\sum_{\mathfrak{c}\in\{2,3,\cdots\}\cup\{\infty\}}\max\{0,\mathfrak{c}-1\}\cdot 0}{1/t}=0\quad(t\to+\infty,\text{ so }t>0\text{ and }\max\{1/t,0\}=1/t)$$

Thus, the measure of discontinuity $\mathcal{D}_{d_1}$ is:

$$\mathcal{D}_0=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\mathcal{M}_j(a,b,d_1,t)\right)\right)=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\mathcal{M}_j(a,b,d_1,t)\right)\right)=\lim_{(a,b)\to\infty}(0)=0$$

This verifies the blockquote in Case 1, which demands $\mathcal{D}_{d_1}=0$

3.3.2. *Case 2.* Suppose $f:X\to Y$ is a function, where $X=\emptyset$, $Y=\mathbb{R}$, $X_1=\mathbb{R}$, $d_1=\dim_{\mathrm{H}}(X_1)=1$ and $\dim_{\mathrm{H}}(X)=0$. Hence, $X'=X\cap X_1=\emptyset$.

> Since Case 2 is an example of Section 2.2.2, criteria 2, the measure of discontinuity $\mathcal{D}_{d_1}(f)$ should be zero.

Similar to Case 1, since $\mathcal{H}^{d_1}(\emptyset)=\mathcal{H}^1(\emptyset)=0$, hence $\mathcal{D}_{d_1}=\mathcal{D}_1=0$
This verifies the blockquote in Case 3.3.2, which demands $\mathcal{D}_{d_1}=0$

3.3.3. *Case 3.* Suppose $f:X\to Y$ is a function, where X is non-empty and finite, $Y=\mathbb{R}$, $X=X_1$, $d_1=\dim_{\mathrm{H}}(X_1)$ and $\dim_{\mathrm{H}}(X)=0$. Hence, $X'=X\cap X_1=X$

> Since Case 3 is an example of Section 2.2.2, criteria 3, the measure of discontinuity $\mathcal{D}_{d_1}(f)$ should be positive infinity.

For parts 1-5, we compute the components $z(\varepsilon,X'(a,b))$, $r(\varepsilon,X'(a,b))$, $h(\varepsilon,X'(a,b))$, $\mathtt{C}_0(d_1,\ell,X'(a,b))$ and $\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))$ (i.e., their motivation is in Section 3.1, page 4), then use these components to compute the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (part 6).

Part 1. $z(\varepsilon, X'(a,b))$

Suppose:

$$z(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & \mathcal{H}^{d_1}(X'(a,b)) = 0,\, \varepsilon > 0 \\ 1/\varepsilon & 0 < \mathcal{H}^{d_1}(X'(a,b)) \leq +\infty,\, \varepsilon > 0 \end{cases}$$

since $0 < \mathcal{H}^0(X'(a,b)) \leq +\infty$ (i.e., the counting measure is $\#|\cdot|$ and $\#|X'(a,b)| > 0$), hence $z(\varepsilon, X'(a,b)) = 1/\varepsilon$.

Part 2. $r(\varepsilon, X'(a,b))$

Suppose:

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^{d_1}(X_j(a,b))\}$$

Note, $X_j(a,b) = X'(a,b)$ (when $X'(a,b)$ is finite) or a sequence discrete finite sets (i.e., $\{X_j(a,b)\}_{j\in\mathbb{N}}$), where $\liminf_{j\to\infty} X_j(a,b) = \limsup_{j\to\infty} X_j(a,b) = X'(a,b)$ (i.e., the set-theoretic limit[15]) and for all $j \in \mathbb{N}$, $0 < \mathcal{H}^0(X_j(a,b)) < +\infty$

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap X_j(a,b)) = \#|X'(a,b)|\}$$

Hence, $\dim_{\mathrm{H}}(\mathrm{Range}(f)) = 0$, since X is discrete and the range of f is discrete. In addition, R can be any set, where the counting measure $\#|R| \geq \#|X_j(a,b)|$ so $\#|X_j(a,b)| \leq \mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) \leq +\infty$. Thus, the smallest $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R)$ can be is $\#|X_j(a,b)|$. This means $\mathbf{R}(X_j(a,b)) = \#|X_j(a,b)|$, where $0 < \#|X_j(a,b)| < +\infty$.

Therefore, $\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = |\#|X_j(a,b)| - \#|X_j(a,b)|| = 0$, for all $j \in \mathbb{N}$ and $0 \leq \lim_{j\to\infty}\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = 0 < +\infty$. Thus, using:

$$r(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & 0 \leq \limsup\limits_{j\to\infty}\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| < +\infty,\, \varepsilon > 0 \\ 1/\varepsilon & \limsup\limits_{j\to\infty}\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = +\infty,\, \varepsilon > 0 \end{cases}$$

we have $r(\varepsilon, X'(a,b)) = \varepsilon$.

Part 3. $h(\varepsilon, X'(a,b))$

Suppose $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.
Notice that $G(X'(a,b))$ is discrete, since $X'(a,b)$ is discrete, so $\dim_{\mathrm{H}}(G(X'(a,b))) = d'(G(X'(a,b))) = 0$. Thus, when $\#|\cdot|$ is the counting measure:

$$\begin{aligned} \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) &= \mathcal{H}^0(G(X'(a,b))) \\ &= \begin{cases} \#|X'(a,b)| & 0 < \#|X'(a,b)| < +\infty \\ +\infty & \#|X'(a,b)| = +\infty \end{cases} \end{aligned}$$

since $f|_{X'(a,b)}$ is discrete, its graph is discrete. Hence, there exists a sequence of finite sets $\{\mathsf{G}_j(X'(a,b))\}_{j\in\mathbb{N}}$, where $\inf_{j\to\infty} \mathsf{G}_j(X'(a,b)) = \inf_{j\to\infty} \mathsf{G}_j(X'(a,b)) = G(X'(a,b))$ and $0 < \mathcal{H}^0(\mathsf{G}_j(X'(a,b))) < +\infty$ for all $j \in \mathbb{N}$. Thus, since $(\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty))$ and $\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0$, then using:

$$h(\varepsilon, X'(a,b)) = \begin{cases} 1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = 0,\, \varepsilon > 0 \\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty)), \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0 \end{cases}$$

we have $h(\varepsilon, X'(a,b)) = 1$

[15] See Section 4.5, page 33

Part 4. $\mathtt{C}_0(d_1,\ell,X'(a,b))$

Suppose, $d_1=0$ and $\#|\cdot|$ is the counting measure.
Again, note that $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.

In addition, when defining:

$$\mathcal{P}_k(d_1,G(X'(a,b)))=\left\{\mathbf{G}\subseteq G(X'(a,b)):\mathcal{H}^{d_1}(\mathbf{G})=k\right\}$$

$$G_k(d_1,X'(a,b))\in\mathcal{P}_k(d_1,G(X'(a,b)))$$

since $\mathbb{G}_0(d_1,X'(a,b))$ is the set of all limit points of $G(X'(a,b))\setminus G_0(d_1,X'(a,b))$ where $G(X'(a,b))$ is discrete, $\mathbf{G}$ is discrete, and $G_0(d_1,X'(a,b))$ is discrete, hence $\mathbb{G}_0(d_1,X'(a,b))$ is empty.

Moreover, $\ell(X'(a,b))\subset\mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$ and since $X'(a,b)$ is non empty, then $\ell(X'(a,b))$ can exist.

Hence, whenever:

$$\mathbf{C}(X)=\begin{cases}1/\varepsilon & \#|X|=0,\varepsilon>0\\ \#|X| & 0<\#|X|<+\infty\\ \varepsilon & \#|X|=+\infty,\ \varepsilon>0\end{cases}\tag{10}$$

since

$$\mathtt{C}_0(d_1,\ell,X'(a,b))=\inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))}\mathbf{C}(\mathbb{G}_0(d_1,X'(a,b))\cap\ell(X'(a,b))$$

$$\inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))}\mathbf{C}(\emptyset\cap\ell(X'(a,b)))=\mathbf{C}(\emptyset)$$

w.r.t. the counting measure $\#|\cdot|$ and Equation 10, $\#|\emptyset|=0$ and $\mathbf{C}(\emptyset)=0$. Thus, $\mathtt{C}_0(d_1,\ell,X'(a,b))=1/\varepsilon$.

Part 5. $\lim_{\varepsilon\to0}\mathbf{M}=\lim_{\varepsilon\to0}(z\cdot r\cdot h\cdot\mathtt{C}_0)$

In Case 3 in Section 3.3.3 Part 1, 2, 3, and 4, where:

- $z(\varepsilon,X'(a,b))=1/\varepsilon$
- $r(\varepsilon,X'(a,b))=\varepsilon$
- $h(\varepsilon,X'(a,b))=1$
- $\mathtt{C}_0(d_1,\ell,X'(a,b))=1/\varepsilon$

$$\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))=\tag{11}$$

$$\lim_{\varepsilon\to0}\left(z(\varepsilon,X'(a,b))\cdot r(\varepsilon,X'(a,b))\cdot h(\varepsilon,X'(a,b))\cdot\mathtt{C}_0(d_1,\ell,X'(a,b))\right)=\lim_{\varepsilon\to0}\left(1/\varepsilon\cdot\varepsilon\cdot1\cdot1/\varepsilon\right)=+\infty\tag{12}$$

Part 6. **Applying The Measure of Discontinuity to** f

Since, for every $B\subseteq X'(a,b)$, $\mathfrak{c}=+\infty$ (see Equation 11):

$$\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=+\infty=\mathfrak{c}$$

Hence, when $\mathfrak{c}=+\infty$ and $B=\mathbf{X}_j(a,b)$, such that:

$$\limsup_{j\to\infty}\mathbf{X}_j(a,b)=\liminf_{j\to\infty}\mathbf{X}_j(a,b)=X'(a,b)$$

and:

$$0<\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b))<+\infty$$

we have $\mathcal{H}^{\dim_{\mathrm{H}}(X')}(B)=\mathcal{H}^0(B)=\#|\mathbf{X}_j(a,b)|$ is the largest possible value. In addition, when $0\le\mathfrak{c}<+\infty$, $\mathcal{H}^0(B)=\mathcal{H}^0(\emptyset)=0$. Therefore:

$$\mathcal{M}(a,b,d_1,t)=\tag{13}$$

$$\left(\frac{\sum\limits_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\lim\limits_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^{\dim_H(X')}(\mathbf{X}_j(a,b))\}}\right)=$$

$$\frac{\sum_{\mathfrak{c}=0}^{\infty}\max\{\mathfrak{c},\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}}+$$
$$\frac{\max\{0,\min\{\infty-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b):\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=+\infty\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}}=$$
$$\frac{\sum_{\mathfrak{c}=1}^{\infty}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^0(\emptyset)}{\max\{1/t,\mathcal{H}^0(\emptyset)\}}+\frac{\max\{0,\min\{+\infty,t\}\}\cdot\#|\mathbf{X}_j(a,b)|}{\max\{1/t,\#|\mathbf{X}(a,b)|\}}=$$
$$\frac{\sum_{\mathfrak{c}=1}^{\infty}\max\{0,\mathfrak{c}-1\}\cdot 0}{\max\{1/t,\#|\mathbf{X}_j(a,b)|\}}+\frac{\max\{0,t\}\cdot\#|\mathbf{X}_j(a,b)|}{\max\{1/t,\#|\mathbf{X}_j(a,b)|\}}=$$
$$\frac{0}{\#|\mathbf{X}_j(a,b)|}+\frac{t\cdot\#|\mathbf{X}_j(a,b)|}{\#|\mathbf{X}_j(a,b)|}=t$$

Thus, the measure of discontinuity $\mathcal{D}_{d_1}$ is:

$$\mathcal{D}_0=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right)=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right)=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}t\right)=+\infty.$$

This verifies the blockquote at the top of Case 3, which states the measure of discontinuity $D_0(f)$ should be positive infinity.

3.3.4. *Case 4.* Suppose $f:X\to Y$ is a function, where $X=\mathbb{R}$, $Y=\mathbb{R}$, $X_1=\mathbb{Q}$, $d_1=\dim_{\mathrm{H}}(X_1)=0$, and $\dim_{\mathrm{H}}(X)=1$. Hence, $X'=X\cap X_1=\mathbb{Q}$.

We define a family of sets $\mathscr{X}=\{\mathsf{X}_c:r\in\{1,\cdots,\mathbf{c_1}\}\}$ where the sets in the family $\mathscr{X}$ are pairwise disjoint, $\bigcup_{r=1}^{\mathbf{c_1}}\mathsf{X}_r=X$ and for all $r\in\{1,\cdots,m\}$, $\mathcal{H}^1(\mathsf{X}_r)=+\infty$ such that the closure[16] of the graph of $f_r:\mathsf{X}_r\to\mathbb{R}$ is continuous on a positive measure subset $\mathbb{X}\subseteq X_1$:[17]

$$f(x)=\begin{cases}f_1(x) & x\in\mathsf{X}_1\\ f_2(x) & x\in\mathsf{X}_2\\ \vdots & \vdots\\ f_{\mathbf{c_1}}(x) & x\in\mathsf{X}_{\mathbf{c_1}}\end{cases}$$

Note, this function can exist [6].

> In addition, Case 4 is an example of Section 2.2.2, criteria 6a so the measure of discontinuity $\mathcal{D}_{d_1}(f)$ needs to be $\mathcal{D}(f)$ (Equation 1, page 3) where $0\le\mathcal{D}_{d_1}(f)\le\mathbf{c_1}-1$.
> See Section 4.6, page 33, for an explicit example.

For parts 1-5, we compute the components $z(\varepsilon,X'(a,b))$, $r(\varepsilon,X'(a,b))$, $h(\varepsilon,X'(a,b))$, $\mathtt{C}_0(d_1,\ell,X'(a,b))$ and $\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))$ (i.e., their motivation is in Section 3.1, page 4), then use these components to compute the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (part 6).

Part 1. $z(\varepsilon,X'(a,b))$

Suppose:

$$z(\varepsilon,X'(a,b))=\begin{cases}\varepsilon & \mathcal{H}^{d_1}(X'(a,b))=0,\,\varepsilon>0\\ 1/\varepsilon & 0<\mathcal{H}^{d_1}(X'(a,b))\le+\infty,\,\varepsilon>0\end{cases}$$

since $0<\mathcal{H}^0(X'(a,b))\le+\infty$ (i.e., $\mathcal{H}^0(\mathbb{Q}\cap(a,b))=+\infty$), hence $z(\varepsilon,X'(a,b))=1/\varepsilon$.

[16] topological closure

[17] See Section 4.1, page 33, where $\mathcal{H}^{\dim_{\mathrm{H}}(X_1)}(\mathbb{X})>0$, $X=\mathsf{X}_r$, $X_1=\mathbb{X}$, and $f(x)=f_r(x)$

Part 2. $r(\varepsilon, X'(a,b))$

Suppose:

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^{d_1}(X_j(a,b))\}$$

since $X'(a,b)$ is countably infinite, there exists a sequence of discrete finite sets (i.e., $\{X_j(a,b)\}_{j\in\mathbb{N}}$), where $\liminf_{j\to\infty} X_j(a,b) = \limsup_{j\to\infty} X_j(a,b) = X'(a,b)$ (i.e., the set-theoretic limit[18]) and for all $j \in \mathbb{N}$, $0 < \mathcal{H}^0(X_j(a,b)) < +\infty$

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap X_j(a,b)) = \#|X_j(a,b)|\}$$

Hence, $\dim_{\mathrm{H}}(\mathrm{Range}(f)) = 0$, since both $X_j(a,b)$ and the range of $f|_{X_j(a,b)}$ are discrete. In addition, $d_1 = 0$. Since $0 < \#|X_j(a,b)| < +\infty$, thereby $\#|R| \geq \#|X_j(a,b)|$ and $\#|X_j(a,b)| \leq \mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) \leq +\infty$. Thus, the smallest $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R)$ can be is $\#|X_j(a,b)|$. That means $\mathbf{R}(X_j(a,b)) = \#|X_j(a,b)|$, where $0 < \#|X_j(a,b)| < +\infty$.

Therefore, $\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = |\#|X_j(a,b)| - \#|X_j(a,b)|| = 0$, for all $j \in \mathbb{N}$ and $0 \leq \lim_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = 0 < +\infty$. Hence, using:

$$r(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & 0 \leq \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| < +\infty,\ \varepsilon > 0 \\ 1/\varepsilon & \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = +\infty,\ \varepsilon > 0 \end{cases}$$

we have $r(\varepsilon, X'(a,b)) = \varepsilon$.

Part 3. $h(\varepsilon, X'(a,b))$

Suppose $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.
Notice that $G(X'(a,b))$ is countably infinite, because $X'(a,b)$ is countably infinite, so $\dim_{\mathrm{H}}(G(X'(a,b))) = d'(G(X'(a,b))) = 0$. Thus, when $\#|\cdot|$ is the counting measure:

$$\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = \mathcal{H}^0(G(X'(a,b))) = +\infty$$

Since $f|_{X'(a,b)}$ is countably infinite, there exists a sequence of finite sets $\{\mathsf{G}_j(X'(a,b))\}_{j\in\mathbb{N}}$, where $\inf_{j\to\infty} \mathsf{G}_j(X'(a,b)) = \inf_{j\to\infty} \mathsf{G}_j(X'(a,b)) = G(X'(a,b))$ and $0 < \mathcal{H}^0(\mathsf{G}_j(X'(a,b))) < +\infty$ for all $j \in \mathbb{N}$

Therefore, since $(\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty))$ and $\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0$, using:

$$h(\varepsilon, X'(a,b)) = \begin{cases} 1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = 0,\ \varepsilon > 0 \\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty)), \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0 \end{cases}$$

we have $h(\varepsilon, X'(a,b)) = 1$

Part 4. $\mathtt{C}_0(d_1, \ell, X'(a,b))$

Suppose, $d_1 = 0$ and $\#|\cdot|$ is the counting measure.
Moreover, $X'(a,b) = \mathbb{Q} \cap (a,b)$ and $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.

Then, note:

$$\mathcal{P}_k(d_1, G(X'(a,b))) = \left\{\mathbf{G} \subseteq G(X'(a,b)) : \mathcal{H}^{d_1}(\mathbf{G}) = k\right\}$$

$$G_k(d_1, X'(a,b)) \in \mathcal{P}_k(d_1, G(X'(a,b)))$$

where $\mathbb{G}_0(d_1, X'(a,b))$ is the set of all limit points of $G(X'(a,b)) \setminus G_0(d_1, X'(a,b))$. Since $\dim_{\mathrm{H}}(G(X'(a,b))) = 0$, there is no $\mathbf{G}$ such that $\mathcal{H}^0(G) = 0$. Therefore, $G_0(d_1, X'(a,b))$ is the empty set. Also, since $X'(a,b) = \mathbb{Q} \cap (a,b)$, $\mathbb{G}_0(d_1, X'(a,b))$ is non-empty and intersects any vertical line with an x-intercept in $\mathbb{R}$.

Moreover, $\ell(X'(a,b)) \subset \mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$ and since $X'(a,b)$ is non-empty, $\ell(X'(a,b))$ can exist.

[18] See Section 4.5, page 33

Hence, whenever:

$$\mathbf{C}(X)=\begin{cases}1/\varepsilon & \#|X|=0,\varepsilon>0\\ \#|X| & 0<\#|X|<+\infty\\ \varepsilon & \#|X|=+\infty,\,\varepsilon>0\end{cases} \tag{14}$$

since

$$\mathtt{C}_0(d_1,\ell,X'(a,b))=\inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))}\mathbf{C}(\mathbb{G}_0(d_1,X'(a,b))\cap\ell(X'(a,b))$$
$$\inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))}\mathbf{C}(\mathbb{G}(X'(a,b))\cap\ell(X'(a,b)))$$

w.r.t. the counting measure $\#|\cdot|$ and Equation 14, the number of times a vertical line intersects with the closure of $\mathbf{c_1}$ functions continuous on positive measure subsets of $X'(a,b)=(a,b)$ is $c=m$ (i.e., $m\geq 0$) and $c=n$ (i.e., $n\leq\mathbf{c_1}$) times. Hence, $0\leq m\leq c=\#|\mathbb{G}(X'(a,b))\cap\ell(X'(a,b))|\leq n\leq\mathbf{c_1}$ and $m\leq c=\mathbf{C}(\mathbb{G}(X'(a,b))\cap\ell(X'(a,b)))\leq n$ so $m\leq c\leq n$.

Part 5. $\lim_{\varepsilon\to0}\mathbf{M}=\lim_{\varepsilon\to0}(z\cdot r\cdot h\cdot\mathtt{C}_0)$

In Case 4 of Section 3.3.4 Part 1, 2, 3, and 4, where:

- $z(\varepsilon,X'(a,b))=1/\varepsilon$
- $r(\varepsilon,X'(a,b))=\varepsilon$
- $h(\varepsilon,X'(a,b))=1$
- $\mathtt{C}_0(d_1,\ell,X'(a,b))=c$, where $m\leq c\leq n$

$$\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))= \tag{15}$$
$$\lim_{\varepsilon\to0}\left(z(\varepsilon,X'(a,b))\cdot r(\varepsilon,X'(a,b))\cdot h(\varepsilon,X'(a,b))\cdot\mathtt{C}_0(d_1,\ell,X'(a,b))\right)=\lim_{\varepsilon\to0}\left(1/\varepsilon\cdot\varepsilon\cdot1\cdot c\right)=c \tag{16}$$

Part 6. **Applying The Measure of Discontinuity to** f

Since, for every $B=B_{\mathfrak{c}}\subseteq X'(a,b)$, $\mathfrak{c}\in\{0\}\cup\mathbb{N}\cup\{+\infty\}$:

$$\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B_{\mathfrak{c}})=\mathfrak{c}$$

Thus, when $\mathfrak{c}=c\in\{m,\cdots,n\}$ and $B_c=B\cap\mathbf{X}_{j,c}(a,b)$, such that:

$$\bigcup_{c=m}^{n}\mathbf{X}_{j,c}(a,b)=\mathbf{X}_j(a,b)$$

where:

$$\limsup_{j\to\infty}\mathbf{X}_j(a,b)=\liminf_{j\to\infty}\mathbf{X}_j(a,b)=X'(a,b)$$

and:

$$0<\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b))<+\infty$$

then

$$\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathfrak{B}_{j,c}(a,b))=\sup\Big(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\Big(\Big\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\,\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=c\Big\}\Big)\Big) \tag{17}$$

is the largest possible value. In addition, when $\mathfrak{c}<m$ or $\mathfrak{c}>n$, then $\mathcal{H}^{\dim_{\mathrm{H}}(X')}(B)=\mathcal{H}^1(\emptyset)=0$ is the largest possible value. Hence:

$$\mathcal{M}(a,b,d_1,t)= \tag{18}$$
$$\left(\frac{\sum_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\Big(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\Big(\Big\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\Big\}\Big)\Big)}{\min\{1/t,\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b))\}}\right)=$$
$$\frac{\sum_{\mathfrak{c}=0}^{m}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\Big(\mathcal{H}^0\Big(\Big\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\Big\}\Big)\Big)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}}+$$

$$\frac{\sum_{\mathfrak{c}=m}^{n}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b):\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}}+$$

$$\frac{\sum_{c_2\in\{n,n+1,\cdots\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b):\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(\mathbf{X}_j(a,b))\}}=$$

$$\frac{\sum_{\mathfrak{c}=0}^{m}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^0(\emptyset)}{\max\{1/t,\mathcal{H}^0(\mathbf{X}_j(a,b))\}}+\frac{\sum_{\mathfrak{c}=m}^{n}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^0(\mathfrak{B}_{j,c}(a,b))}{\max\{1/t,\mathcal{H}^0(\mathbf{X}_j(a,b))\}}+$$

$$\frac{\sum_{\mathfrak{c}\in\{n,n+1,\cdots\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^0(\emptyset)}{\max\{1/t,\mathcal{H}^0(\mathbf{X}_j(a,b))\}}=$$

$$\frac{\sum_{\mathfrak{c}=0}^{m}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot 0}{\max\{1/t,\mathcal{H}^0(\mathbf{X}_j(a,b))\}}+\frac{\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^0(\mathfrak{B}_{j,c}(a,b))}{\max\{1/t,\mathcal{H}^0(\mathbf{X}_j(a,b))\}}+\frac{\sum_{\mathfrak{c}\in\{n,n+1,\cdots\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot 0}{\max\{1/t,\mathcal{H}^0(\mathbf{X}_j(a,b))\}}=$$

$$\frac{\sum_{\mathfrak{c}=m}^{n}\max\{0,\mathfrak{c}-1\}\cdot\mathcal{H}^0(\mathfrak{B}_{j,c}(a,b))}{\mathcal{H}^0(\mathbf{X}_j(a,b))}=$$

$$\frac{\sum_{\mathfrak{c}=m}^{n}(\mathfrak{c}-1)\cdot\#|\mathfrak{B}_{j,c}(a,b)|}{\#|\mathbf{X}_j(a,b)|}$$

Thus, the measure of discontinuity $\mathcal{D}_0(f)$ is:

$$\begin{aligned}\mathcal{D}_0(f)&=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right)\\&=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right)\\&=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\frac{\sum_{\mathfrak{c}=m}^{n}(\mathfrak{c}-1)\cdot\#|\mathfrak{B}_{j,c}(a,b)|}{\#|\mathbf{X}_j(a,b)|}\right)\right)=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\frac{\sum_{\mathfrak{c}=m}^{n}(\mathfrak{c}-1)\cdot\#|\mathfrak{B}_{j,c}(a,b)|}{\#|\mathbf{X}_j(a,b)|}\right)\right)\end{aligned}\tag{19}$$

Also, because $\mathfrak{B}_{j,c}(a,b)\subseteq\mathbf{X}_j(a,b)$ and $0\le m\le c\le n\le\mathbf{c_1}$,

$$\begin{aligned}0=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\frac{\max\{0,\min\{0-1,t\}\}\cdot\#|\mathfrak{B}_{j,0}(a,b)|}{\#|\mathbf{X}_j(a,b)|}\right)\right)&\le\mathcal{D}_0(f)\\\le\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\frac{(\mathbf{c_1}-1)\cdot\#|\mathfrak{B}_{j,\mathbf{c_1}}(a,b)|}{\#|\mathbf{X}_j(a,b)|}\right)\right)&=\mathbf{c_1}-1\end{aligned}\tag{20}$$

since $\mathfrak{B}_{j,0}=\mathbf{X_j}(a,b)$ and $\max\{0,\min\{0-1,t\}\}=0$. In addition, $\mathfrak{B}_{j,\mathbf{c_1}}=\mathbf{X}_j(a,b)$ and $\max\{0,\min\{\mathbf{c_1}-1,t\}\}=\mathbf{c_1}-1$, when $t\ge\mathbf{c_1}-1$.

Moreover, in Section 2.2.2 criteria 6a, note that $\mathcal{D}_0(f)=\mathcal{D}(f)$ (Equation 38) where the definition of $\mathcal{D}(f)$ is defined with this following list:

(1) the variable $\mathfrak{c}$ is the number of the times the vertical line intersects the closure[19] with respect to its x-intercept
(2) $\limsup_{j\to\infty}\mathbf{X}'_j=\liminf_{j\to\infty}\mathbf{X}'_j=X'$ (i.e., the set theoretic limit[20]) such that $0<\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}'_j)<+\infty$ for all $j\in\mathbb{N}$
(3) the arbitrary set $\overline{X_{\mathfrak{c}}}\subseteq X\cap X_1$ has the largest Hausdorff measure in its dimension, such that the vertical line for all $x\in\overline{X_{\mathfrak{c}}}$ intersects the closure $\mathfrak{c}$ times ($m\le\mathfrak{c}\le n$)

$$\mathcal{D}(f)=\lim_{(a,b)\to(-\infty,\infty)}\left(\liminf_{j\to\infty}\frac{\sum_{\mathfrak{c}=m}^{n}(\mathfrak{c}-1)\cdot\mathcal{H}^0(\mathbf{X}'_j\cap\overline{X_{\mathfrak{c}}}\cap(a,b))}{\mathcal{H}^0((X\cap\mathbf{X}'_j\cap X_1)\cap(a,b))\}}\right)\tag{21}$$

[19] topological closure
[20] See Section 4.5, page 33

$$= \lim_{(a,b)\to(-\infty,\infty)} \left(\limsup_{j\to\infty} \frac{\sum_{\mathfrak{c}=m}^{n} (\mathfrak{c}-1)\cdot\mathcal{H}^0(\mathbf{X}'_j \cap \overline{X_{\mathfrak{c}}} \cap (a,b))}{\mathcal{H}^0((X\cap\mathbf{X}'_j\cap X_1)\cap(a,b))} \right) \tag{22}$$

Hence, $\mathcal{D}_0(f) = \mathcal{D}(f)$, since:

- $\limsup_{j\to\infty} \mathbf{X}'_j \cap (a,b) = \liminf_{j\to\infty} \mathbf{X}'_j \cap (a,b) = X'(a,b)$ (i.e., the set-theoretic limit[20]) and $\limsup_{j\to\infty} \mathbf{X}_j(a,b) = \liminf_{j\to\infty} \mathbf{X}_j(a,b) = X'(a,b)$
- $\limsup_{j\to\infty} \mathbf{X}'_j \cap \overline{X_{\mathfrak{c}}} \cap (a,b) = \liminf_{j\to\infty} \mathbf{X}'_j \cap \overline{X_{\mathfrak{c}}} \cap (a,b) = \limsup_{j\to\infty} \mathfrak{B}_{j,c}(a,b) = \liminf_{j\to\infty} \mathfrak{B}_{j,c}(a,b)$ (i.e., see the definition of $\mathfrak{B}_{j,c}(a,b)$ on page 13 and the definitions of the sets in the left hand side of the equality on page 14, criteria 1-3)
- $\limsup_{j\to\infty}(X\cap\mathbf{X}'_j\cap X_1)\cap(a,b) = \liminf_{j\to\infty}(X\cap\mathbf{X}'_j\cap X_1)\cap(a,b) = \limsup_{j\to\infty}\mathbf{X}_j(a,b) = \liminf_{j\to\infty}\mathbf{X}_j(a,b)$

Thus, with the bulleted list, we proved $\mathcal{D}_0(f) = \mathcal{D}(f)$ and verified the blockquote at the top of Case 4, which states that the measure of discontinuity $D_0(f)$ is between integers 0 and $\mathbf{c_1}$.

3.3.5. *Case 5.* Suppose $f : X \to Y$ is a function, where $X = \mathbb{Q}$, $Y = \mathbb{R}$, $d_1 = \dim_{\mathrm{H}}(X_1) = 1$, and $\dim_{\mathrm{H}}(X) = 0$. Hence, $X' = X \cap X_1 = \mathbb{Q}$. (Notice, $d_1 > \dim_{\mathrm{H}}(X)$.)

We define a family of sets $\mathscr{X} = \{\mathsf{X}_r : r \in \{1,\cdots,\mathbf{c_1}\}\}$ where the sets in the family $\mathscr{X}$ are pairwise disjoint, $\bigcup_{m=1}^{\mathbf{c_1}} \mathsf{X}_r = X$, and for all $r \in \{1,\cdots,\mathbf{c_1}\}$; $\mathcal{H}^0(\mathsf{X}_r) = +\infty$ such that the closure[19] of $f_r : \mathsf{X}_r \to \mathbb{R}$ is continuous on the positive measure subset $\mathbb{X} \subseteq X_1$:[21]

$$f(x) = \begin{cases} f_1(x) & x \in \mathsf{X}_1 \\ f_2(x) & x \in \mathsf{X}_2 \\ \vdots & \vdots \\ f_{\mathfrak{c}}(x) & x \in \mathsf{X}_m \end{cases}$$

Note, this function can exist: e.g.,

$$\mathsf{X}_r = \begin{cases} \left\{ s_1/\left(2^{(\mathbf{c_1}-1)}t_1\right) : s_1, t_1 \in \mathbb{N} \right\} & \mathbf{c_1} = 1 \\ \left\{ s_r/\left(2^{(\mathbf{c_1}-r)}t_r\right) : s_r \in \text{odd}\,\mathbb{N},\, t_r \in \text{odd}\,\mathbb{N} \right\} \setminus \{\mathsf{X}_1\} & 2 \le r \le \mathbf{c_1} \end{cases} \tag{23}$$

where we prove the sets in $\mathscr{X} = \{\mathsf{X}_r : r \in \{1,\cdots,\mathbf{c}_1\}\}$ are pairwise disjoint using Mathematical Induction.[22]

> In addition, Case 5 is an example of Section 2.2.2, criteria 6c, so the measure of discontinuity $\mathcal{D}_{d_1}(f)$ should be positive infinity. (See Section 4.7, page 34, for an explicit example.)

For parts 1-5, we compute the components $z(\varepsilon, X'(a,b))$, $r(\varepsilon, X'(a,b))$, $h(\varepsilon, X'(a,b))$, $\mathsf{C}_0(d_1,\ell,X'(a,b))$ and $\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))$ (i.e., their motivation is in Section 3.1, page 4), then use these components to compute the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (part 6).

Part 1. $z(\varepsilon, X'(a,b))$

Suppose:

$$z(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & \mathcal{H}^{d_1}(X'(a,b)) = 0,\, \varepsilon > 0 \\ 1/\varepsilon & 0 < \mathcal{H}^{d_1}(X'(a,b)) \le +\infty,\, \varepsilon > 0 \end{cases}$$

since $\mathcal{H}^1(X'(a,b)) = 0$ (i.e., $\mathcal{H}^1(\mathbb{Q}\cap(a,b)) = 0$), thus $z(\varepsilon, X'(a,b)) = \varepsilon$.

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[21] See Section 4.1, page 33, $\mathcal{H}^{\dim_{\mathrm{H}}(X_1)}(\mathbb{X}) > 0$, $X = \mathsf{X}_r$, $X_1 = \mathbb{X}$, and $f(x) = f_r(x)$

[22] See Section 4.8, page 34.

Part 2. $r(\varepsilon, X'(a,b))$

Suppose:

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^{d_1}(X_j(a,b))\}$$

Note, $X_j(a,b)$ exists (i.e., $1 = d_1 > \dim_{\mathrm{H}}(X) = 0$) so there exists a sequence $\{X_j(a,b)\}_{j\in\mathbb{N}}$ where both $\liminf_{j\to\infty} X_j(a,b) = \limsup_{j\to\infty} X_j(a,b) = X'(a,b) = \mathbb{Q} \cap (a,b)$ and for all $j \in \mathbb{N}$, $0 < \mathcal{H}^1(X_j(a,b)) < +\infty$ (e.g., $\{q_j\}_{j\in\mathbb{N}}$ is an enumeration of the rationals, ϵ approaches 0, and $X_j(a,b) = [q_j - \epsilon/2^j, q_j + \epsilon/2^j]$.)

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{1}(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^{1}(X_j(a,b))\}$$

Hence, $\dim_{\mathrm{H}}(\mathrm{Range}(f)) = 0$, since the graph of f is countably infinite. Note, R is any subset of $f[\mathbb{Q}]$, where $\mathcal{H}^1(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^1(X_j(a,b))$; however, since $\mathcal{H}^1(X_j(a,b)) > 0$ and $\mathcal{H}^1(f^{-1}[R] \cap X_j(a,b)) \leq \mathcal{H}^1(f^{-1}[f[\mathbb{Q}]] \cap X_j(a,b)) = 0 \neq \mathcal{H}^1(X_j(a,b))$. Hence, $R = \emptyset$ and $\inf(\emptyset) = +\infty$.

Therefore, $|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))| = |(+\infty) - \mathcal{H}^{d_1}(X_j(a,b))| = +\infty$, for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} |\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))| = +\infty$. Thus, using:

$$r(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & 0 \leq \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| < +\infty,\ \varepsilon > 0 \\ 1/\varepsilon & \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = +\infty,\ \varepsilon > 0 \end{cases}$$

we have $r(\varepsilon, X'(a,b)) = 1/\varepsilon$.

Part 3. $h(\varepsilon, X'(a,b))$

Suppose $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.
Since $X'(a,b)$ is countably infinite, $G(X'(a,b))$ is countably infinite. Thus, $d'(G(X'(a,b)) = 0$ and $\mathcal{H}^{d'(G(X'(a,b))}(G(X'(a,b))) = +\infty$.

Moreover, $\{\mathsf{G}_j(X'(a,b))\}_{j\in\mathbb{N}}$ exists: that is, consider the enumeration $\{g_n\}_{n\in\mathbb{N}}$ of $G(X'(a,b))$, such that $\mathsf{G}_j(X'(a,b)) = (g_n + \varepsilon/2^j, g_n + \varepsilon/2^j)$.

Hence, using:

$$h(\varepsilon, X'(a,b)) = \begin{cases} 1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = 0,\ \varepsilon > 0 \\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty)), \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0 \end{cases}$$

we have $h(\varepsilon, X'(a,b)) = 1$

Part 4. $\mathsf{C}_0(d_1, \ell, X'(a,b))$

Suppose, $d_1 = 1$ and $\#|\cdot|$ is the counting measure.
Again, note that $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.

In addition, note that:

$$\mathcal{P}_k(d_1, G(X'(a,b))) = \left\{\mathbf{G} \subseteq G(X'(a,b)) : \mathcal{H}^{d_1}(\mathbf{G}) = k\right\}$$

$$G_k(d_1, X'(a,b)) \in \mathcal{P}_k(d_1, G(X'(a,b)))$$

since $\mathbb{G}_0(d_1, X'(a,b))$ is the set of all limit points of $G(X'(a,b)) \setminus G_0(d_1, X'(a,b))$ and $d_1 = 1$, hence for all sets $\mathbf{G} \subseteq G$, note that $\mathcal{H}^1(\mathbf{G}) = 0$. Thus, $G_0(d_1, X'(a,b)) = G(X'(a,b))$ and $\mathbb{G}_0(d_1, X'(a,b)) = G(X'(a,b)) \setminus G_0(d_1, X'(a,b)) = \emptyset$.

Moreover, $\ell(X'(a,b)) \subset \mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$ and since $X'(a,b)$ is non empty, $\ell(X'(a,b))$ can exist.

Hence, whenever:

$$\mathbf{C}(X) = \begin{cases} 1/\varepsilon & \#|X| = 0, \varepsilon > 0 \\ \#|X| & 0 < \#|X| < +\infty \\ \varepsilon & \#|X| = +\infty,\ \varepsilon > 0 \end{cases} \tag{24}$$

$$\mathsf{C}_0(d_1, \ell, X'(a,b)) = \inf_{G_0(d_1, X'(a,b)) \in \mathcal{P}_0(d_1, X'(a,b))} \mathbf{C}(\mathbb{G}_0(d_1, X'(a,b)) \cap \ell(X'(a,b))$$

$$= \inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))} \mathbf{C}(\emptyset\cap\ell(X'(a,b)))$$
$$= \inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))} \mathbf{C}(\emptyset)$$

w.r.t. counting measure $\#|\cdot|$ and Equation 24, $\#|\emptyset| = 0$ and $\mathbf{C}(\emptyset) = 1/\varepsilon$

Part 5. $\lim_{\varepsilon\to 0}\mathbf{M} = \lim_{\varepsilon\to 0}(z\cdot r\cdot h\cdot \mathtt{C}_0)$

In Case 5 of Section 3.3.5 Part 1, 2, 3, and 4, where:

- $z(\varepsilon, X'(a,b)) = \varepsilon$
- $r(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $h(\varepsilon, X'(a,b)) = 1$
- $\mathtt{C}_0(d_1,\ell, X'(a,b)) = 1/\varepsilon$

$$\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon, d_1, \ell, X'(a,b)) = \tag{25}$$
$$\lim_{\varepsilon\to 0}\left(z(\varepsilon,X'(a,b))\cdot r(\varepsilon,X'(a,b))\cdot h(\varepsilon,X'(a,b))\cdot \mathtt{C}_0(d_1,\ell,X'(a,b))\right) = \lim_{\varepsilon\to 0}\left(\varepsilon\cdot 1/\varepsilon\cdot 1\cdot 1/\varepsilon\right) = +\infty \tag{26}$$

Part 6. **Applying The Measure of Discontinuity to** f

Since, for every $B\subseteq X'(a,b)$, $\mathfrak{c} = +\infty$ (see Equation 25):

$$\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon, d_1, \ell, B) = +\infty = \mathfrak{c}$$

Hence, when $\mathfrak{c} = +\infty$ and $B = X'(a,b)$, such that:

$$\limsup_{j\to\infty}\mathbf{X}_j(a,b) = \liminf_{j\to\infty}\mathbf{X}_j(a,b) = X'(a,b)$$

and:

$$0 < \mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b)) < +\infty$$

$\mathcal{H}^{\dim_{\mathrm{H}}(X')}(B) = \mathcal{H}^0(X'(a,b)) = \mathcal{H}^0(\mathbb{Q}\cap(a,b))$ so $\mathcal{H}^0(\mathbf{X}_j(a,b)) = \#|\mathbf{X}_j(a,b)|$ is the largest possible value. Also, when $0\le\mathfrak{c}<+\infty$, then $\mathcal{H}^{\dim_{\mathrm{H}}(X')}(B) = \mathcal{H}^1(\emptyset) = 0$ is the largest possible value. Hence:

$$\mathcal{M}(a,b,d_1,t) = \tag{27}$$
$$\left(\frac{\sum_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X)}\left(\left\{B\cap\mathbf{X}_j(a,b) : B\subseteq X'(a,b), \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^{\dim_{\mathrm{H}}(X)}(\mathbf{X}_j(a,b))\}}\right) =$$
$$\frac{\sum_{\mathfrak{c}\in\{0\}\cup\mathbb{N}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b) : B\subseteq X'(a,b), \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}} +$$
$$\frac{\max\{0,\min\{+\infty-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b) : B\subseteq X'(a,b) : \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=+\infty\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}} =$$
$$\frac{\sum_{\mathfrak{c}\in\{0\}\cup\mathbb{N}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^0(\emptyset)}{\max\{1/t,\#|X_j(a,b)|\}} + \frac{\max\{0,\min\{+\infty-1,t\}\}\cdot\mathcal{H}^0(\mathbf{X}_j(a,b))}{\max\{1/t,\#|X_j(a,b)|\}} =$$
$$\frac{\sum_{\mathfrak{c}\in\{0,\cdots,t\}}\max\{0,t\}\cdot 0}{\max\{1/t,\#|X_j(a,b)|\}} + \frac{\sum_{\mathfrak{c}\in\{t,\cdots,\mathfrak{c}-1\}}\max\{0,\mathfrak{c}-1\}\cdot 0}{\max\{1/t,\#|X_j(a,b)|\}} + \frac{\max\{0,t\}\cdot\#|X_j(a,b)|}{\#|X_j(a,b)|} = \tag{28}$$
$$\frac{\sum_{\mathfrak{c}\in\{0,\cdots,t\}} t\cdot 0}{\#|X_j(a,b)|} + \frac{\sum_{\mathfrak{c}\in\{t,\cdots,\mathfrak{c}-1\}}(\mathfrak{c}-1)\cdot 0}{\#|X_j(a,b)|} + \frac{t\cdot\#|X_j(a,b)|}{\#|X_j(a,b)|} = t \tag{29}$$

Thus, the measure of discontinuity $\mathcal{D}_{d_1}$ is:

$$\mathcal{D}_0 = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty} t\right) = +\infty$$

This verifies the blockquote at the top of Case 5, which states the measure of discontinuity $\mathcal{D}_{d_1}(f) = +\infty$ should be positive infinity.

3.3.6. *Case 6.* Suppose $f : X \to Y$ is a function, $X = \mathbb{R}$, $Y = \mathbb{R}$, $X_1 = \mathbb{R}$, $d_1 = 1$, $\dim_{\mathrm{H}}(X) = 1$. Hence, $X \cap X_1 = \mathbb{R}$.

We define a family of sets $\mathscr{X} = \{\mathsf{X}_c : r \in \{1, \cdots, \mathbf{c_1}\}\}$ where the sets in the family $\mathscr{X}$ are pairwise disjoint, $\bigcup_{r=1}^{\mathbf{c_1}} \mathsf{X}_r = X$ and for all $r \in \{1, \cdots, m\}$, $\mathcal{H}^1(\mathsf{X}_r) = +\infty$ such that the closure[23] of the graph of $f_r : \mathsf{X}_r \to \mathbb{R}$ is continuous on a positive measure subset $\mathbb{X} \subseteq X_1$:[24]

$$f(x) = \begin{cases} f_1(x) & x \in \mathsf{X}_1 \\ f_2(x) & x \in \mathsf{X}_2 \\ \vdots & \vdots \\ f_{\mathbf{c_1}}(x) & x \in \mathsf{X}_{\mathbf{c_1}} \end{cases}$$

Note, this function can exist [6].

In addition, Case 6 is an example of Section 2.2.2, criteria 6a, so the measure of discontinuity $\mathcal{D}_{d_1}(f)$ needs to be $\mathcal{D}(f)$ (Equation 1, page 3) where $0 \le \mathcal{D}_{d_1}(f) \le \mathbf{c_1} - 1$.

See Section 4.6, page 33, for an explicit example.

For parts 1-5, we compute the components $z(\varepsilon, X'(a,b))$, $r(\varepsilon, X'(a,b))$, $h(\varepsilon, X'(a,b))$, $\mathsf{C}_0(d_1, \ell, X'(a,b))$ and $\lim_{\varepsilon\to 0} \mathbf{M}(\varepsilon, d_1, \ell, X'(a,b))$ (i.e., their motivation is in Section 3.1, page 4), then use these components to compute the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (part 6).

Part 1. $z(\varepsilon, X'(a,b))$

Suppose:

$$z(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & \mathcal{H}^{d_1}(X'(a,b)) = 0,\, \varepsilon > 0 \\ 1/\varepsilon & 0 < \mathcal{H}^{d_1}(X'(a,b)) \le +\infty,\, \varepsilon > 0 \end{cases}$$

since $0 < \mathcal{H}^1(X'(a,b)) \le +\infty$ (i.e., $\mathcal{H}^1(\mathbb{R} \cap (a,b)) = b - a$), hence $z(\varepsilon, X'(a,b)) = 1/\varepsilon$.

Part 2. $r(\varepsilon, X'(a,b))$

Suppose:

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^{d_1}(X_j(a,b))\}$$

since $X'(a,b) = (a,b)$, there exists a sequence of sets (i.e., $\{X_j(a,b)\}_{j\in\mathbb{N}}$ where for all $j \in \mathbb{N}$, $X_j(a,b) = X'(a,b)$), $\liminf_{j\to\infty} X_j(a,b) = \limsup_{j\to\infty} X_j(a,b) = X'(a,b)$ (i.e., the set-theoretic limit[25]), and for all $j \in \mathbb{N}$, $0 < \mathcal{H}^1(X_j(a,b)) = \mathcal{H}^1(X'(a,b)) = b - a < +\infty$.

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R \subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R] \cap X_j(a,b)) = \#|X_j(a,b)|\}$$

Thus, $0 \le \dim_{\mathrm{H}}(\mathrm{Range}(f)) \le 1$, since $\dim_{\mathrm{H}}(\mathrm{Range}(f)) \le \dim_{\mathrm{H}}(X_j(a,b)) = \dim_{\mathrm{H}}(X'(a,b)) = \dim_{\mathrm{H}}((a,b)) = 1$. In addition, $d_1 = 1$. Since $0 < \mathcal{H}^{d_1}(X_j(a,b)) = \mathcal{H}^1(X'(a,b)) = \mathcal{H}^1(a,b) = b - a < +\infty$, hence $\mathcal{H}^1(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^1(f^{-1}[f[X_j(a,b)]] \cap X_j(a,b)) = \mathcal{H}^1(X_j(a,b)) = b - a$ so $f[X_j(a,b)] \subseteq R$ and the smallest $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R)$ can be is $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(f[X_j(a,b)])$. That means $\mathbf{R}(X_j(a,b)) = \mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(f[X_j(a,b)])$, where $0 < \mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(f[X_j(a,b)]) < +\infty$, so there exists a constant $C > 0$ such that $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(f[X_j(a,b)]) = C \cdot \mathcal{H}^{d_1}(X_j(a,b))$

Therefore, $\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = |C \cdot \mathcal{H}^1(X_j(a,b)) - \mathcal{H}^1(X_j(a,b))| = |(C-1)\mathcal{H}^1(X_j(a,b))| = (|C-1|)\mathcal{H}^1(X_j(a,b))$, for all $j \in \mathbb{N}$ and $0 \le |(C-1)\mathcal{H}^1(X_j(a,b))| < +\infty$. Hence, using:

$$r(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & 0 \le \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| < +\infty,\, \varepsilon > 0 \\ 1/\varepsilon & \limsup\limits_{j\to\infty} \left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = +\infty,\, \varepsilon > 0 \end{cases}$$

we have $r(\varepsilon, X'(a,b)) = \varepsilon$.

[23]topological closure

[24]See Section 4.1, page 33, where $\mathcal{H}^{\dim_{\mathrm{H}}(X_1)}(\mathbb{X}) > 0$, $X = \mathsf{X}_r$, $X_1 = \mathbb{X}$, and $f(x) = f_r(x)$

[25]See Section 4.5, page 33

Part 3. $h(\varepsilon, X'(a,b))$

Suppose $G(X'(a,b))$ is the graph of $f|_{X'(a,b)} = f|_{(a,b)}$.
Note that:

$$\mathcal{H}^{\dim_{\mathrm{H}}(G(X'(a,b)))}(G(X'(a,b)) = \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b)) = \mathcal{H}^1(G(X'(a,b)) = \sum_{m=1}^{\mathbf{c}_1} \mathcal{H}^1(\mathsf{X}_m)\text{Arclength}(\text{graph}(f_m|_{(a,b)})) \tag{30}$$

which is finite, since the arclength of $\mathbf{c}_1$ functions on a finite interval is finite.

Since $d_1 = \dim_{\mathrm{H}}(G(X'(a,b)) = 1$ and $\mathcal{H}^{\dim_{\mathrm{H}}(G(X'(a,b)))}(G(X'(a,b))$ is finite, there then exists a sequence of finite sets $\{\mathsf{G}_j(X'(a,b))\}_{j\in\mathbb{N}}$, where $\inf_{j\to\infty}\mathsf{G}_j(X'(a,b)) = \inf_{j\to\infty}\mathsf{G}_j(X'(a,b)) = G(X'(a,b))$ and $0 < \mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) < +\infty$ for all $j\in\mathbb{N}$.

Thus, since $(\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty))$ and $\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0$, using:

$$h(\varepsilon, X'(a,b)) = \begin{cases} 1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = 0,\, \varepsilon > 0 \\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty)), \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0 \end{cases}$$

we have $h(\varepsilon, X'(a,b)) = 1$

Part 4. $\mathtt{C}_0(d_1, \ell, X'(a,b))$

Suppose, $d_1 = 0$ and $\#|\cdot|$ is the counting measure.
In addition, $X'(a,b) = (a,b)$ and $G(X'(a,b))$ is the graph of $f|_{X'(a,b)} = f|_{(a,b)}$.

Thus, note that:

$$\mathcal{P}_k(d_1, G(X'(a,b))) = \left\{\mathbf{G} \subseteq G(X'(a,b)) : \mathcal{H}^{d_1}(\mathbf{G}) = k\right\}$$
$$G_k(d_1, X'(a,b)) \in \mathcal{P}_k(d_1, G(X'(a,b)))$$

and $\mathbb{G}_0(d_1, X'(a,b))$ is the set of all limit points of $G(X'(a,b)) \setminus G_0(d_1, X'(a,b))$. Since $\dim_{\mathrm{H}}(G(X'(a,b))) = 1$, there exists $\mathbf{G}$, where $\mathcal{H}^1(G) = 0$. However, in this case, it is unnecessary to remove any $G_0(d_1, X'(a,b))$ from $G(X'(a,b))$, since the graph of f is nowhere dense in $\mathbb{R}^2$ and removing any $G_0(d_1, X'(a,b))$ from $G(X'(a,b))$ will not affect the maximum number of times an arbitrary vertical line intersects the closure of $G(X'(a,b))$. Therefore, $\mathbb{G}_0(d_1, X'(a,b))$ intersects any vertical line with an x-intercept in $\mathbb{R}$.

Moreover, $\ell(X'(a,b)) \subset \mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$ and since $X'(a,b)$ is non-empty, $\ell(X'(a,b))$ can exist.
Hence, whenever:

$$\mathbf{C}(X) = \begin{cases} 1/\varepsilon & \#|X| = 0, \varepsilon > 0 \\ \#|X| & 0 < \#|X| < +\infty \\ \varepsilon & \#|X| = +\infty,\, \varepsilon > 0 \end{cases} \tag{31}$$

since

$$\mathtt{C}_0(d_1, \ell, X'(a,b)) = \inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))} \mathbf{C}(\mathbb{G}_0(d_1, X'(a,b)) \cap \ell(X'(a,b))$$
$$\inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))} \mathbf{C}(\mathbb{G}(X'(a,b)) \cap \ell(X'(a,b)))$$

w.r.t. the counting measure $\#|\cdot|$ and Equation 31, the number of times a vertical line intersects with the closure of $\mathbf{c_1}$ functions continuous on positive measure subsets of $X'(a,b) = (a,b)$ is $c = m$ (i.e., $m \geq 0$) and $c = n$ (i.e., $n \leq \mathbf{c_1}$) times. Hence, $0 \leq m \leq c = \#|\mathbb{G}(X'(a,b)) \cap \ell(X'(a,b))| \leq n \leq \mathbf{c_1}$ and $m \leq c = \mathbf{C}(\mathbb{G}(X'(a,b)) \cap \ell(X'(a,b))) \leq n$ so $m \leq c \leq n$.

Part 5. $\lim_{\varepsilon\to 0}\mathbf{M} = \lim_{\varepsilon\to 0}(z\cdot r\cdot h\cdot \mathtt{C}_0)$

In Case 6 of Section 3.3.6 Part 1, 2, 3, and 4, where:

- $z(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $r(\varepsilon, X'(a,b)) = \varepsilon$
- $h(\varepsilon, X'(a,b)) = 1$

- $\mathtt{C}_0(d_1,\ell,X'(a,b))=c$, where $m\le c\le n$

$$\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))= \tag{32}$$
$$\lim_{\varepsilon\to0}\left(z(\varepsilon,X'(a,b))\cdot r(\varepsilon,X'(a,b))\cdot h(\varepsilon,X'(a,b))\cdot \mathtt{C}_0(d_1,\ell,X'(a,b))\right)=\lim_{\varepsilon\to0}\left(1/\varepsilon\cdot\varepsilon\cdot1\cdot c\right)=c \tag{33}$$

Part 6. Applying The Measure of Discontinuity to f

Since, for every $B=B_{\mathfrak{c}}\subseteq X'(a,b)$, $\mathfrak{c}\in\{0\}\cup\mathbb{N}\cup\{+\infty\}$:

$$\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B_{\mathfrak{c}})=\mathfrak{c}$$

Thus, when $\mathfrak{c}=c\in\{m,\cdots,n\}$ and $B_c=B\cap\mathbf{X}_{j,c}(a,b)$, such that:

$$\bigcup_{c=m}^{n}\mathbf{X}_{j,c}(a,b)=\mathbf{X}_j(a,b)$$

where:

$$\limsup_{j\to\infty}\mathbf{X}_j(a,b)=\liminf_{j\to\infty}\mathbf{X}_j(a,b)=X'(a,b)$$

and:

$$0<\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b))<+\infty$$

then

$$\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathfrak{B}_{j,c}(a,b))=\sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\ \lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=c\right\}\right)\right) \tag{34}$$

is the largest possible value. In addition, when $\mathfrak{c}<m$ or $\mathfrak{c}>n$, then $\mathcal{H}^{\dim_{\mathrm{H}}(X')}(B)=\mathcal{H}^1(\emptyset)=0$ is the largest possible value. Hence:

$$\mathcal{M}(a,b,d_1,t)= \tag{35}$$
$$\left(\frac{\sum\limits_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\lim\limits_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b))\}}\right)=$$
$$\frac{\sum\limits_{\mathfrak{c}=0}^{m}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^1\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\lim\limits_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^1(X_j(a,b))\}}+$$
$$\frac{\sum\limits_{\mathfrak{c}=m}^{n}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^1\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b):\lim\limits_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^1(X_j(a,b))\}}+$$
$$\frac{\sum\limits_{c_2\in\{n,n+1,\cdots\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^1\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b):\lim\limits_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^1(\mathbf{X}_j(a,b))\}}=$$
$$\frac{\sum_{\mathfrak{c}=0}^{m}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^1(\emptyset)}{\max\{1/t,\mathcal{H}^1(\mathbf{X}_j(a,b))\}}+\frac{\sum\limits_{\mathfrak{c}=m}^{n}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^1(\mathfrak{B}_{j,c}(a,b))}{\max\{1/t,\mathcal{H}^1(\mathbf{X}_j(a,b))\}}+$$
$$\frac{\sum_{\mathfrak{c}\in\{n,n+1,\cdots\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^1(\emptyset)}{\max\{1/t,\mathcal{H}^1(\mathbf{X}_j(a,b))\}}=$$
$$\frac{\sum_{\mathfrak{c}=0}^{m}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot0}{\max\{1/t,\mathcal{H}^1(\mathbf{X}_j(a,b))\}}+\frac{\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^1(\mathfrak{B}_{j,c}(a,b))}{\max\{1/t,\mathcal{H}^1(\mathbf{X}_j(a,b))\}}+\frac{\sum_{\mathfrak{c}\in\{n,n+1,\cdots\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot0}{\max\{1/t,\mathcal{H}^1(\mathbf{X}_j(a,b))\}}=$$
$$\frac{\sum\limits_{\mathfrak{c}=m}^{n}\max\{0,\mathfrak{c}-1\}\cdot\mathcal{H}^1(\mathfrak{B}_{j,c}(a,b))}{\mathcal{H}^1(\mathbf{X}_j(a,b))}=$$
$$\frac{\sum\limits_{\mathfrak{c}=m}^{n}(\mathfrak{c}-1)\cdot\#|\mathfrak{B}_{j,c}(a,b)|}{\#|\mathbf{X}_j(a,b)|}$$

Thus, the measure of discontinuity $\mathcal{D}_0(f)$ is:

$$\mathcal{D}_0(f) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right)$$
$$= \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right)$$
$$= \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\frac{\sum_{\mathfrak{c}=m}^{n}(\mathfrak{c}-1)\cdot\#|\mathfrak{B}_{j,c}(a,b)|}{\#|\mathbf{X}_j(a,b)|}\right)\right) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\frac{\sum_{\mathfrak{c}=m}^{n}(\mathfrak{c}-1)\cdot\#|\mathfrak{B}_{j,c}(a,b)|}{\#|\mathbf{X}_j(a,b)|}\right)\right) \tag{36}$$

Also, because $\mathfrak{B}_{j,c}(a,b)\subseteq\mathbf{X}_j(a,b)$ and $0\le m\le c\le n\le\mathbf{c_1}$,

$$0 = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\frac{\max\{0,\min\{0-1,t\}\}\cdot\#|\mathfrak{B}_{j,0}(a,b)|}{\#|\mathbf{X}_j(a,b)|}\right)\right)\le\mathcal{D}_0(f) \tag{37}$$
$$\le \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\frac{(\mathbf{c_1}-1)\cdot\#|\mathfrak{B}_{j,\mathbf{c_1}}(a,b)|}{\#|\mathbf{X}_j(a,b)|}\right)\right)=\mathbf{c_1}-1$$

since $\mathfrak{B}_{j,0}=\mathbf{X_j}(a,b)$ and $\max\{0,\min\{0-1,t\}\}=0$. In addition, $\mathfrak{B}_{j,\mathbf{c_1}}=\mathbf{X}_j(a,b)$ and $\max\{0,\min\{\mathbf{c_1}-1,t\}\}=\mathbf{c_1}-1$, when $t\ge\mathbf{c_1}-1$.

Moreover, in Section 2.2.2 criteria 6a, note that $\mathcal{D}_0(f)=\mathcal{D}(f)$ (Equation 38) where the definition of $\mathcal{D}(f)$ is defined with this following list:

(1) the variable $\mathfrak{c}$ is the number of the times the vertical line intersects the closure[26] with respect to its x-intercept
(2) $\limsup_{j\to\infty}\mathbf{X}'_j=\liminf_{j\to\infty}\mathbf{X}'_j=X'$ (i.e., the set theoretic limit[27]) such that $0<\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}'_j)<+\infty$ for all $j\in\mathbb{N}$
(3) the arbitrary set $\overline{X_{\mathfrak{c}}}\subseteq X\cap X_1$ has the largest Hausdorff measure in its dimension, such that the vertical line for all $x\in\overline{X_{\mathfrak{c}}}$ intersects the closure $\mathfrak{c}$ times ($m\le\mathfrak{c}\le n$)

$$\mathcal{D}(f) = \lim_{(a,b)\to(-\infty,\infty)}\left(\liminf_{j\to\infty}\frac{\sum_{\mathfrak{c}=m}^{n}(\mathfrak{c}-1)\cdot\mathcal{H}^1(\mathbf{X}'_j\cap\overline{X_{\mathfrak{c}}}\cap(a,b))}{\mathcal{H}^1((X\cap\mathbf{X}'_j\cap X_1)\cap(a,b))\}}\right) \tag{38}$$
$$= \lim_{(a,b)\to(-\infty,\infty)}\left(\limsup_{j\to\infty}\frac{\sum_{\mathfrak{c}=m}^{n}(\mathfrak{c}-1)\cdot\mathcal{H}^1(\mathbf{X}'_j\cap\overline{X_{\mathfrak{c}}}\cap(a,b))}{\mathcal{H}^1((X\cap\mathbf{X}'_j\cap X_1)\cap(a,b))}\right) \tag{39}$$

Hence, $\mathcal{D}_0(f)=\mathcal{D}(f)$, since:

- $\limsup_{j\to\infty}\mathbf{X}'_j\cap(a,b)=\liminf_{j\to\infty}\mathbf{X}'_j\cap(a,b)=X'(a,b)$ (i.e., the set-theoretic limit[27]) and $\limsup_{j\to\infty}\mathbf{X}_j(a,b)=\liminf_{j\to\infty}\mathbf{X}_j(a,b)=X'(a,b)$
- $\limsup_{j\to\infty}\mathbf{X}'_j\cap\overline{X_{\mathfrak{c}}}\cap(a,b)=\liminf_{j\to\infty}\mathbf{X}'_j\cap\overline{X_{\mathfrak{c}}}\cap(a,b)=\limsup_{j\to\infty}\mathfrak{B}_{j,c}(a,b)=\liminf_{j\to\infty}\mathfrak{B}_{j,c}(a,b)$ (i.e., see the definition of $\mathfrak{B}_{j,c}(a,b)$ on page 20 and the definitions of the sets in the left hand side of the equality on page 21, criteria 1-3)
- $\limsup_{j\to\infty}(X\cap\mathbf{X}'_j\cap X_1)\cap(a,b)=\liminf_{j\to\infty}(X\cap\mathbf{X}'_j\cap X_1)\cap(a,b)=\limsup_{j\to\infty}\mathbf{X}_j(a,b)=\liminf_{j\to\infty}\mathbf{X}_j(a,b)$

Thus, with the bulleted list, we proved $\mathcal{D}_1(f)=\mathcal{D}(f)$ and verified the blockquote at the top of Case 4, which states that the measure of discontinuity $D_0(f)$ is between integers 0 and $\mathbf{c_1}$.

3.3.7. *Case 7.* Suppose $f:X\to Y$ is a function, where $X=\mathbb{Q}$, $Y=\mathbb{R}$, $X_1=\mathbb{Q}$, $d_1=\dim_{\mathrm{H}}(X_1)=0$, and $\dim_{\mathrm{H}}(X)=0$. Hence, $X'=X\cap X_1=\mathbb{Q}$.

We define a family of sets $\mathscr{X}=\{\mathsf{X}_c:r\in\{1,\cdots,\mathbf{c_1}\}\}$ where the sets in the family $\mathscr{X}$ are pairwise disjoint, $\bigcup_{r=1}^{\mathbf{c_1}}\mathsf{X}_r=X$ and for all $r\in\{1,\cdots,m\}$, $\mathcal{H}^1(\mathsf{X}_r)=+\infty$ such that the closure[28] of the graph of $f_r:\mathsf{X}_r\to\mathbb{R}$ is continuous on a positive measure subset $\mathbb{X}\subseteq X_1$:[29]

[26] topological closure
[27] See Section 4.5, page 33
[28] topological closure
[29] See Section 4.1, page 33, where $\mathcal{H}^{\dim_{\mathrm{H}}(X_1)}(\mathbb{X})>0$, $X=\mathsf{X}_r$, $X_1=\mathbb{X}$, and $f(x)=f_r(x)$

$$f(x)=\begin{cases} f_1(x) & x\in \mathsf{X}_1\\ f_2(x) & x\in \mathsf{X}_2\\ \vdots & \vdots \\ f_{\mathbf{c_1}}(x) & x\in \mathsf{X}_{\mathbf{c_1}}\end{cases}$$

Note, this function can exist [6].

> In addition, Case 7 is an example of Section 2.2.2, criteria 6b so the measure of discontinuity $\mathcal{D}_{d_1}(f)$ needs to be $\mathcal{D}(f)$ (Equation 1, page 3) where $0\le \mathcal{D}_{d_1}(f)\le \mathbf{c_1}-1$.
> See Section 4.6, page 33, for an explicit example.

Since $d_1=0$ and $X'=X\cap X_1$, the computations in Case 7 are exactly the same as Case 4.

Therefore, similar to Case 4, we proved $\mathcal{D}_0(f)=\mathcal{D}(f)$ and verified the blockquote at the top of Case 7 which states that the measure of discontinuity $D_0(f)$ is between integers 0 and $\mathbf{c_1}$.

3.3.8. *Case 8.* Suppose $f: X\to Y$ is a function, where $X=\mathbb{Q}\cap[0,1]$, $Y=\mathbb{R}$, $X_1=\mathbb{Q}\cap[0,1]$, $d_1=\dim_{\mathrm{H}}(X_1)=0$, and $\dim_{\mathrm{H}}(X)=0$. Hence, $X'=X\cap X_1=\mathbb{Q}\cap[0,1]$.

Therefore, $f(p/q)=1/q$ for all coprime integers $p,q\in\mathbb{Z}$. Note that f is hyper-discontinuous.[30]

> In addition, Case 8 is an example of Section 2.2.2, criteria 5 so the measure of discontinuity $\mathcal{D}_{d_1}(f)$ needs to be positive infinity.

For parts 1-5, we compute the components $z(\varepsilon,X'(a,b))$, $r(\varepsilon,X'(a,b))$, $h(\varepsilon,X'(a,b))$, $\mathtt{C}_0(d_1,\ell,X'(a,b))$ and $\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))$ (i.e., their motivation is in Section 3.1, page 4), then use these components to compute the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (part 6).

Part 1. $z(\varepsilon,X'(a,b))$

Suppose:

$$z(\varepsilon,X'(a,b))=\begin{cases}\varepsilon & \mathcal{H}^{d_1}(X'(a,b))=0,\ \varepsilon>0\\ 1/\varepsilon & 0<\mathcal{H}^{d_1}(X'(a,b))\le+\infty,\ \varepsilon>0\end{cases}$$

since $0<\mathcal{H}^0(X'(a,b))\le+\infty$ (i.e., $\mathcal{H}^0(\mathbb{Q}\cap(a,b))=+\infty$, when $(a,b)\subset[0,1]$ and $[0,1]\subseteq(a,b)$), hence $z(\varepsilon,X'(a,b))=1/\varepsilon$. For simplicity, we say $(a,b)\supset[0,1]$ such that $(a,b)\to[0,1]$.

Part 2. $r(\varepsilon,X'(a,b))$

Suppose:

$$\mathbf{R}(X_j(a,b))=\inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R): R\subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R]\cap X_j(a,b))=\mathcal{H}^{d_1}(X_j(a,b))\}$$

since $X'(a,b)$ is countably infinite, there exists a sequence of discrete finite sets (i.e., $\{X_j(a,b)\}_{j\in\mathbb{N}}$), where $\liminf_{j\to\infty}X_j(a,b)=\limsup_{j\to\infty}X_j(a,b)=X'(a,b)$ (i.e., the set-theoretic limit[31]) and for all $j\in\mathbb{N}$, $0<\mathcal{H}^0(X_j(a,b))<+\infty$

$$\mathbf{R}(X_j(a,b))=\inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R): R\subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R]\cap X_j(a,b))=\#|X_j(a,b)|\}$$

Hence, $\dim_{\mathrm{H}}(\mathrm{Range}(f))=0$, since $X_j(a,b)$ and the range of $f|_{X_j(a,b)}$ are discrete. In addition, $d_1=0$. Thus, since $0<\#|X_j(a,b)|<+\infty$, thereby $\#|R|\ge\#|X_j(a,b)|$ and $\#|X_j(a,b)|\le\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R)\le+\infty$; however, there exists a lower bound greater than $\#|X_j(a,b)|$ since $\mathrm{Range}(f)=\{1/q: q\in\mathbb{N}\}$ and for all $q\in\mathbb{N}$, there exist $\phi(q)$ isolated points on $y=1/q$ such that $\phi(\cdot)$ is the Euler's Totient Function.

To find the second lower bound, we note v is a natural number, $R=\{1,1/2,\cdots,1/v\}$, and $\mathcal{H}^{d_1}(f^{-1}[\{1,1/2,\cdots,1/v\}])=\mathcal{H}^0(f^{-1}[\{1,1/2,\cdots,1/v\}])=\sum_{i=1}^{v}\phi(v)=\frac{3}{\pi^2}v^2+O(v\log v)$ where O is the Big-O notation and $\sum_{i=1}^{v}\phi(v)$ is the Totient summatory function. This is required since $f[X_j(a,b)]$ must cover the entire range of f and the set-theoretic limit[31] of $X_j(a,b)$ is $\mathbb{Q}\cap[0,1]$.

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[30] See Section 4.3, page 33

[31] See Section 4.5, page 33

Moreover, $\mathcal{H}^{d_1}(X_j(a,b)) = \mathcal{H}^0(X_j(a,b)) = \#|X_j(a,b)|$, $R = \{1, 1/2, \ldots, 1/v\}$, and $\mathcal{H}^0(f^{-1}[R] \cap X_j(a,b)) = \mathcal{H}^0(X_j(a,b))$, hence $\mathcal{H}^0(f^{-1}[\{1,\cdots,v\}] \cap X_j(a,b)) = \mathcal{H}^0(X_j(a,b)) = \#|X_j(a,b)| \leq \mathcal{H}^0(f^{-1}[\{1,\cdots,1/v\}])$. Thus, since $\mathcal{H}^0(f^{-1}[\{1,\cdots,1/v\}]) = \frac{3}{\pi^2}v^2 + O(x\log x)$ and $\mathcal{H}^0(f^{-1}[\{1,\cdots,v\}] \cap X_j(a,b)) = (1/C)\cdot\mathcal{H}^0(f^{-1}[\{1,\cdots,1/v\}])$ for some constant $C > 0$, we have $\frac{1}{C}\frac{3}{\pi^2}v^2 < \frac{1}{C}\mathcal{H}^0(f^{-1}[\{1,\cdots,1/v\}]) = \mathcal{H}^0(X_j(a,b)) = \#|X_j(a,b)| < \frac{1}{C}\frac{3}{\pi^2}(v+1)^2$. Next, we solve v, by the following:

$$\begin{aligned}
&\frac{1}{C}\frac{3}{\pi^2}v^2 < \mathcal{H}^0(X_j(a,b)) = \#|X_j(a,b)| \\
&\frac{1}{C}\frac{3}{\pi^2}v^2 < \#|X_j(a,b)| \\
&v^2 < \frac{C\pi^2}{3}\#|X_j(a,b)| \\
&v = \left\lfloor \sqrt{\frac{C\pi^2}{3}\#|X_j(a,b)|} \right\rfloor
\end{aligned}$$

Hence, the smallest $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R)$ can be is:

$$\mathcal{H}^0(R) = \mathcal{H}^0\left(\left\{0,\cdots,\frac{1}{v}\right\}\right) = v = \left\lfloor \sqrt{\frac{C\pi^2}{3}\#|X_j(a,b)|} \right\rfloor.$$

That means:

$$\mathbf{R}(X_j(a,b)) = \left\lfloor \sqrt{\frac{C\pi^2}{3}\#|X_j(a,b)|} \right\rfloor.$$

Therefore:

$$\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = \left|\left\lfloor \sqrt{\frac{C\pi^2}{3}\#|X_j(a,b)|} \right\rfloor - \#|X_j(a,b)|\right|$$

and when $x \mapsto \#|X_j(a,b)|$ and $x \to \infty$ (i.e., since $\lim_{j\to\infty}\#|X_j(a,b)| = \infty$)

$$\lim_{x\to\infty}\left|\left\lfloor \sqrt{\frac{C\pi^2}{3}x} \right\rfloor - x\right| = +\infty.$$

Hence, for all $j \in \mathbb{N}$ and $\lim_{j\to\infty}\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = +\infty$. Thus, using:

$$r(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & 0 \leq \limsup_{j\to\infty}\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| < +\infty,\, \varepsilon > 0 \\ 1/\varepsilon & \limsup_{j\to\infty}\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = +\infty,\, \varepsilon > 0 \end{cases}$$

we have $r(\varepsilon, X'(a,b)) = 1/\varepsilon$.

Part 3. $h(\varepsilon, X'(a,b))$

Suppose $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.
Notice that $G(X'(a,b))$ is countably infinite since $X'(a,b$ is countably infinite, so $\dim_{\mathrm{H}}(G(X'(a,b))) = d'(G(X'(0,1))) = 0$. Hence, when $\#|\cdot|$ is the counting measure:

$$\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = \mathcal{H}^0(G(X'(a,b))) = +\infty$$

Since $f|_{X'(a,b)}$ is countably infinite, there exists a sequence of finite sets $\{\mathsf{G}_j(X'(a,b))\}_{j\in\mathbb{N}}$, where $\inf_{j\to\infty}\mathsf{G}_j(X'(a,b)) = \inf_{j\to\infty}\mathsf{G}_j(X'(a,b)) = G(X'(a,b))$ and $0 < \mathcal{H}^0(\mathsf{G}_j(X'(a,b))) < +\infty$ for all $j \in \mathbb{N}$

Thus, since $(\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty))$ and $\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0$, using:

$$h(\varepsilon, X'(a,b)) = \begin{cases} 1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = 0,\, \varepsilon > 0 \\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty)), \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0 \end{cases}$$

we have $h(\varepsilon, X'(a,b)) = 1$

Part 4. $\mathtt{C}_0(d_1, \ell, X'(a,b))$

Suppose, $d_1 = 0$ and $\#|\cdot|$ is the counting measure.
Moreover, $X'(a,b) = \mathbb{Q} \cap (a,b)$ and $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.
Thus, note:

$$\mathcal{P}_k(d_1, G(X'(a,b))) = \left\{\mathbf{G} \subseteq G(X'(a,b)) : \mathcal{H}^{d_1}(\mathbf{G}) = k\right\}$$
$$G_k(d_1, X'(a,b)) \in \mathcal{P}_k(d_1, G(X'(a,b)))$$

where $\mathbb{G}_0(d_1, X'(a,b))$ is the set of all limit points of $G(X'(a,b)) \setminus G_0(d_1, X'(a,b))$. Since $\dim_{\mathrm{H}}(G(X'(a,b))) = 0$, there exists no $\mathbf{G}$ such that $\mathcal{H}^0(G) = 0$. Therefore, $G_0(d_1, X'(a,b))$ is the empty set. Also, the range of f is $\{1/q : q \in \mathbb{N}\}$ where for all $q \in \mathbb{N}$, there exists $\phi(q)$ points[32] on $y = 1/q$. Since the range of f is dense in $\{0\}$, $\lim_{q\to\infty} \phi(q) = +\infty$ and for all prime q, $\{(p/q, 1/q) : p \in \mathbb{N},\ 0 < p < q\}$ is evenly distributed[33] on $y = 1/q$, hence $\mathbb{G}_0(d_1, X'(a,b)) = \{(x, 0) : x \in \mathbb{Q} \cap [0,1]\}$ and any vertical line with an x-intercept in $\mathbb{R}$ intersects once with $\mathbb{G}_0(d_1, X'(a,b))$.

Moreover, $\ell(X'(a,b)) \subset \mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$ and since $X'(a,b)$ is non-empty, $\ell(X'(a,b))$ can exist.
Hence, whenever:

$$\mathbf{C}(X) = \begin{cases} 1/\varepsilon & \#|X| = 0, \varepsilon > 0 \\ \#|X| & 0 < \#|X| < +\infty \\ \varepsilon & \#|X| = +\infty,\ \varepsilon > 0 \end{cases} \tag{40}$$

since

$$\mathtt{C}_0(d_1, \ell, X'(a,b)) = \inf_{G_0(d_1, X'(a,b)) \in \mathcal{P}_0(d_1, X'(a,b))} \mathbf{C}(\mathbb{G}_0(d_1, X'(a,b)) \cap \ell(X'(a,b))$$
$$\inf_{G_0(d_1, X'(a,b)) \in \mathcal{P}_0(d_1, X'(a,b))} \mathbf{C}(\mathbb{G}(X'(a,b)) \cap \ell(X'(a,b)))$$

w.r.t. the counting measure $\#|\cdot|$ and Equation 40, $\#|\mathbb{G}(X'(a,b)) \cap \ell(X'(a,b))| = 1$ and $\mathbf{C}(\mathbb{G}(X'(a,b)) \cap \ell(X'(a,b))) = 1$.

Part 5. $\lim_{\varepsilon\to 0} \mathbf{M} = \lim_{\varepsilon\to 0}(z \cdot r \cdot h \cdot \mathtt{C}_0)$

In Case 8 of Section 3.3.8 Part 1, 2, 3, and 4, where:

- $z(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $r(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $h(\varepsilon, X'(a,b)) = 1$
- $\mathtt{C}_0(d_1, \ell, X'(a,b)) = 1$

$$\lim_{\varepsilon\to 0} \mathbf{M}(\varepsilon, d_1, \ell, X'(a,b)) = \tag{41}$$
$$\lim_{\varepsilon\to 0} (z(\varepsilon, X'(a,b)) \cdot r(\varepsilon, X'(a,b)) \cdot h(\varepsilon, X'(a,b)) \cdot \mathtt{C}_0(d_1, \ell, X'(a,b))) = \lim_{\varepsilon\to 0} (1/\varepsilon \cdot 1/\varepsilon \cdot 1 \cdot c) = +\infty \tag{42}$$

Part 6. **Applying The Measure of Discontinuity to** f

Since, for every $B \subseteq X'(a,b)$, $\mathfrak{c} = +\infty$ (see Equation 41):

$$\lim_{\varepsilon\to 0} \mathbf{M}(\varepsilon, d_1, \ell, B) = +\infty = \mathfrak{c}$$

Hence, when $\mathfrak{c} = +\infty$ and $B = \mathbf{X}_j(a,b)$, such that:

$$\limsup_{j\to\infty} \mathbf{X}_j(a,b) = \liminf_{j\to\infty} \mathbf{X}_j(a,b) = X'(a,b)$$

and:

$$0 < \mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b)) < +\infty$$

[32] $\phi(\cdot)$ is Euler's Totient function (see Section 4.9, page 4.9)
[33] The set of all x-values in $T_q = \{(p/q, 1/q) : p \in \mathbb{N},\ 0 < p < q\}$ or $t_q = \{p/q : 0 < p < q\}$ is Equidistributed on $[\alpha, \beta] = [0,1]$ (see Section 4.10, page 35)

then $\mathcal{H}^{\dim_{\mathrm{H}}(X')}(B) = \mathcal{H}^0(B) = \#|\mathbf{X}_j(a,b)|$ is the largest possible value. Also, when $0 \le \mathfrak{c} < +\infty$, $\mathcal{H}^0(B) = \mathcal{H}^0(\emptyset) = 0$. Therefore:

$$\mathcal{M}(a,b,d_1,t) = \tag{43}$$

$$\left(\frac{\sum_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}} \max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\left(\left\{B\cap\mathbf{X}_j(a,b) : B\subseteq X'(a,b), \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^{\dim_H(X')}(\mathbf{X}_j(a,b))\}}\right) =$$

$$\frac{\sum_{\mathfrak{c}=0}^{\infty}\max\{\mathfrak{c},\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b) : B\subseteq X'(a,b), \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}} +$$

$$\frac{\max\{0,\min\{\infty-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b) : B\subseteq X'(a,b) : \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=+\infty\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}} =$$

$$\frac{\sum_{\mathfrak{c}=1}^{\infty}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^0(\emptyset)}{\max\{1/t,\mathcal{H}^0(\emptyset)\}} + \frac{\max\{0,\min\{+\infty,t\}\}\cdot\#|\mathbf{X}_j(a,b)|}{\max\{1/t,\#|\mathbf{X}(a,b)|\}} =$$

$$\frac{\sum_{\mathfrak{c}=1}^{\infty}\max\{0,\mathfrak{c}-1\}\cdot 0}{\max\{1/t,\#|\mathbf{X}_j(a,b)|\}} + \frac{\max\{0,t\}\cdot\#|\mathbf{X}_j(a,b)|}{\max\{1/t,\#|\mathbf{X}_j(a,b)|\}} =$$

$$\frac{0}{\#|\mathbf{X}_j(a,b)|} + \frac{t\cdot\#|\mathbf{X}_j(a,b)|}{\#|\mathbf{X}_j(a,b)|} = t$$

Hence, the measure of discontinuity $\mathcal{D}_{d_1}$ is:

$$\mathcal{D}_0 = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty} t\right) = +\infty$$

This verifies the blockquote at the top of Case 8, which states the measure of discontinuity $D_0(f)$ should be positive infinity.

3.3.9. *Case 9.* Suppose $f : X \to Y$ is a function, $X = \mathbb{Q}$, $Y = \mathbb{R}$, $X_1 = \mathbb{Q}$, $d_1 = \dim_{\mathrm{H}}(X_1) = 0$, $\dim_{\mathrm{H}}(X) = 0$. Hence, $X' = X \cap X_1 = \mathbb{Q}$.

We define f such that its graph is dense in $\mathbb{R}^2$

Note, this function can exist [3].

> Case 9 is not mentioned in Section 2.2.2; however, we want to see if the measure of discontinuity $\mathcal{D}_{d_1}(f)$ gives an intuitive answer.

For parts 1-5, we compute the components $z(\varepsilon, X'(a,b))$, $r(\varepsilon, X'(a,b))$, $h(\varepsilon, X'(a,b))$, $\mathtt{C}_0(d_1,\ell,X'(a,b))$ and $\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))$ (i.e., their motivation is in Section 3.1, page 4), then use these components to compute the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (part 6).

Part 1. $z(\varepsilon, X'(a,b))$

Suppose:

$$z(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & \mathcal{H}^{d_1}(X'(a,b)) = 0,\ \varepsilon > 0 \\ 1/\varepsilon & 0 < \mathcal{H}^{d_1}(X'(a,b)) \le +\infty,\ \varepsilon > 0 \end{cases}$$

since $0 < \mathcal{H}^0(X'(a,b)) \le +\infty$ (i.e., $\mathcal{H}^0(\mathbb{Q}\cap(a,b)) = +\infty$), hence $z(\varepsilon, X'(a,b)) = 1/\varepsilon$.

Part 2. $r(\varepsilon, X'(a,b))$

Suppose:

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R\subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R]\cap X_j(a,b)) = \mathcal{H}^{d_1}(X_j(a,b))\}$$

since $X'(a,b$ is countably infinite, there exists a sequence of discrete finite sets (i.e., $\{X_j(a,b)\}_{j\in\mathbb{N}}$), where $\liminf_{j\to\infty} X_j(a,b) = \limsup_{j\to\infty} X_j(a,b) = X'(a,b)$ (i.e., the set-theoretic limit[34]) and for all $j\in\mathbb{N}$, $0 < \mathcal{H}^0(X_j(a,b)) < +\infty$

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R\subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R]\cap X_j(a,b)) = \#|X_j(a,b)|\}$$

[34] See Section 4.5, page 33

Hence, $\dim_{\mathrm{H}}(\mathrm{Range}(f))=0$, since $X_j(a,b)$ and the range of $f|_{X_j(a,b)}$ are discrete. In addition, $d_1=0$. Since $0<\#|X_j(a,b)|<+\infty$, thereby $\#|R|\geq\#|X_j(a,b)|$ and $\#|X_j(a,b)|\leq\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R)\leq+\infty$; however, there exists a lower bound greater than $\#|X_j(a,b)|$ because Range(f) is dense in $\mathbb{R}$ and $R=f[X_j(a,b)]$. Hence, since the graph of f is dense in $\mathbb{R}^2$ and $f[X_j(a,b)]$ is dense in $\mathbb{R}$, the smallest $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R)=\mathcal{H}^0(R)$ can be is $+\infty$. That means $\mathbf{R}(X_j(a,b))=+\infty$.

Therefore, $\left|\mathbf{R}(X_j(a,b))-\mathcal{H}^{d_1}(X_j(a,b))\right|=|(+\infty)-\#|X_j(a,b)||=0$, for all $j\in\mathbb{N}$ and $\lim_{j\to\infty}|\mathbf{R}(X_j(a,b))-\mathcal{H}^{d_1}(X_j(a,b))|=+\infty$. Hence, using:

$$r(\varepsilon,X'(a,b))=\begin{cases}\varepsilon & 0\leq\limsup\limits_{j\to\infty}\left|\mathbf{R}(X_j(a,b))-\mathcal{H}^{d_1}(X_j(a,b))\right|<+\infty,\ \varepsilon>0\\ 1/\varepsilon & \limsup\limits_{j\to\infty}\left|\mathbf{R}(X_j(a,b))-\mathcal{H}^{d_1}(X_j(a,b))\right|=+\infty,\ \varepsilon>0\end{cases}$$

we have $r(\varepsilon,X'(a,b))=1/\varepsilon$.

Part 3. $h(\varepsilon,X'(a,b))$

Suppose $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.
Notice that $G(X'(a,b))$ is countably infinite, since $X'(a,b)$ is countably infinite, so $\dim_{\mathrm{H}}(G(X'(a,b)))=d'(G(X'(a,b)))=0$. Thus, when $\#|\cdot|$ is the counting measure:

$$\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b)))=\mathcal{H}^0(G(X'(a,b)))=+\infty$$

Since $f|_{X'(a,b)}$ is countably infinite, there exists a sequence of finite sets $\{\mathsf{G}_j(X'(a,b))\}_{j\in\mathbb{N}}$, where $\inf_{j\to\infty}\mathsf{G}_j(X'(a,b))=\inf_{j\to\infty}\mathsf{G}_j(X'(a,b))=G(X'(a,b))$ and $0<\mathcal{H}^0(\mathsf{G}_j(X'(a,b)))<+\infty$ for all $j\in\mathbb{N}$

Thus, since $(\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b)))\in(0,+\infty))$ and $\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b)))\neq0$, using:

$$h(\varepsilon,X'(a,b))=\begin{cases}1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b)))=0,\ \varepsilon>0\\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b)))\in(0,+\infty)),\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b)))\neq0\end{cases}$$

we have $h(\varepsilon,X'(a,b))=1$

Part 4. $\mathsf{C}_0(d_1,\ell,X'(a,b))$

Suppose, $d_1=0$ and $\#|\cdot|$ is the counting measure.
Moreover, $X'(a,b)=\mathbb{Q}\cap(a,b)$ and $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.

Thus, note:

$$\mathcal{P}_k(d_1,G(X'(a,b)))=\left\{\mathbf{G}\subseteq G(X'(a,b)):\mathcal{H}^{d_1}(\mathbf{G})=k\right\}$$
$$G_k(d_1,X'(a,b))\in\mathcal{P}_k(d_1,G(X'(a,b)))$$

where $\mathbb{G}_0(d_1,X'(a,b))$ is the set of all limit points of $G(X'(a,b))\setminus G_0(d_1,X'(a,b))$. Since $\dim_{\mathrm{H}}(G(X'(a,b)))=0$, there exists no $\mathbf{G}$ such that $\mathcal{H}^0(G)=0$. Therefore, $G_0(d_1,X'(a,b))$ is the empty set. Also, since $X'(a,b)=\mathbb{Q}\cap(a,b)$, $\mathbb{G}_0(d_1,X'(a,b))=\mathbb{R}^2$ and intersects any vertical line with an x-intercept in $\mathbb{R}$.

Moreover, $\ell(X'(a,b))\subset\mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$ and since $X'(a,b)$ is non-empty, $\ell(X'(a,b))$ can exist.
Hence, whenever:

$$\mathbf{C}(X)=\begin{cases}1/\varepsilon & \#|X|=0,\varepsilon>0\\ \#|X| & 0<\#|X|<+\infty\\ \varepsilon & \#|X|=+\infty,\ \varepsilon>0\end{cases}\tag{44}$$

since

$$\begin{aligned}\mathsf{C}_0(d_1,\ell,X'(a,b))=&\inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))}\mathbf{C}(\mathbb{G}_0(d_1,X'(a,b))\cap\ell(X'(a,b))\\ &\inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))}\mathbf{C}(\mathbb{G}(X'(a,b))\cap\ell(X'(a,b)))\end{aligned}$$

w.r.t. the counting measure $\#|\cdot|$ and Equation 44, $\#|\mathbb{G}(d_1,X'(a,b))\cap\ell(X'(a,b))|=+\infty$ and $\mathbf{C}(\mathbb{G}(d_1,X'(a,b))\cap\ell(X'(a,b)))=\varepsilon$

Part 5. $\lim_{\varepsilon\to 0}\mathbf{M} = \lim_{\varepsilon\to 0}(z\cdot r\cdot h\cdot \mathtt{C}_0)$

In Case 9 of Section 3.3.9 Part 1, 2, 3, and 4, where:

- $z(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $r(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $h(\varepsilon, X'(a,b)) = 1$
- $\mathtt{C}_0(d_1,\ell, X'(a,b)) = \varepsilon$

$$\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon, d_1,\ell, X'(a,b)) = \tag{45}$$

$$\lim_{\varepsilon\to 0}\left(z(\varepsilon, X'(a,b))\cdot r(\varepsilon, X'(a,b))\cdot h(\varepsilon, X'(a,b))\cdot \mathtt{C}_0(d_1,\ell, X'(a,b))\right) = \lim_{\varepsilon\to 0}\left(1/\varepsilon\cdot 1/\varepsilon\cdot 1\cdot\varepsilon\right) = +\infty \tag{46}$$

Part 6. **Applying The Measure of Discontinuity to** f

Since, for every $B\subseteq X'(a,b)$, $\mathfrak{c} = +\infty$ (see Equation 49):

$$\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon, d_1,\ell, B) = +\infty = \mathfrak{c}$$

Hence, when $\mathfrak{c} = +\infty$ and $B = \mathbf{X}_j(a,b)$, such that:

$$\limsup_{j\to\infty}\mathbf{X}_j(a,b) = \liminf_{j\to\infty}\mathbf{X}_j(a,b) = X'(a,b)$$

and:

$$0 < \mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b)) < +\infty$$

then $\mathcal{H}^{\dim_{\mathrm{H}}(X')}(B) = \mathcal{H}^0(B) = \#|\mathbf{X}_j(a,b)|$ is the largest possible value. Also, when $0\le\mathfrak{c}<+\infty$, $\mathcal{H}^0(B) = \mathcal{H}^0(\emptyset) = 0$. Hence:

$$\mathcal{M}(a,b,d_1,t) = \tag{47}$$

$$\left(\frac{\sum_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\left(\left\{B\cap\mathbf{X}_j(a,b): B\subseteq X'(a,b), \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B) = \mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^{\dim_H(X')}(\mathbf{X}_j(a,b))\}}\right) =$$

$$\frac{\sum_{\mathfrak{c}=0}^{\infty}\max\{\mathfrak{c},\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b): B\subseteq X'(a,b), \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B) = \mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}} +$$

$$\frac{\max\{0,\min\{\infty-1,t\}\}\cdot\sup\left(\mathcal{H}^0\left(\left\{B\cap\mathbf{X}_j(a,b): B\subseteq X'(a,b): \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B) = +\infty\right\}\right)\right)}{\min\{1/t,\mathcal{H}^0(X_j(a,b))\}} =$$

$$\frac{\sum_{\mathfrak{c}=1}^{\infty}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^0(\emptyset)}{\max\{1/t,\mathcal{H}^0(\emptyset)\}} + \frac{\max\{0,\min\{+\infty,t\}\}\cdot\#|\mathbf{X}_j(a,b)|}{\max\{1/t,\#|\mathbf{X}(a,b)|\}} =$$

$$\frac{\sum_{\mathfrak{c}=1}^{\infty}\max\{0,\mathfrak{c}-1\}\cdot 0}{\max\{1/t,\#|\mathbf{X}_j(a,b)|\}} + \frac{\max\{0,t\}\cdot\#|\mathbf{X}_j(a,b)|}{\max\{1/t,\#|\mathbf{X}_j(a,b)|\}} =$$

$$\frac{0}{\#|\mathbf{X}_j(a,b)|} + \frac{t\cdot\#|\mathbf{X}_j(a,b)|}{\#|\mathbf{X}_j(a,b)|} = t$$

Thus, the measure of discontinuity $\mathcal{D}_{d_1}$ is:

$$\mathcal{D}_0 = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}t\right) = +\infty$$

The measure of discontinuity $D_0(f)$ is intuitive, since the dense subset $D\subseteq\mathrm{graph}(f)$ has $\mathcal{H}^0(D) = +\infty$.

3.3.10. *Case 10.* Suppose $f: X\to Y$ is a function, $X=\mathbb{R}$, $Y=\mathbb{R}$, $X_1=\mathbb{R}$, $d_1 = 1$, $\dim_{\mathrm{H}}(X) = 1$, f is the Conway-Base 13 function [7].

In addition, Case 10 is everywhere surjective[35], but is defined at $f(x) = 0$ for *almost all* $x\in\mathbb{R}$ [4]. This case is an example of Section 2.2.2, criteria 7b, so since $f(x)=0$ for *almost all* $x\in\mathbb{R}$, the measure of discontinuity $\mathcal{D}_{d_1}(f)$ needs to be zero.

[35] See Section 4.4, page 33

For parts 1-5, we compute the components $z(\varepsilon, X'(a,b))$, $r(\varepsilon, X'(a,b))$, $h(\varepsilon, X'(a,b))$, $\mathsf{C}_0(d_1,\ell,X'(a,b))$ and $\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))$ (i.e., their motivation is in Section 3.1, page 4), then use these components to compute the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (part 6).

Part 1. $z(\varepsilon, X'(a,b))$

Suppose:

$$z(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & \mathcal{H}^{d_1}(X'(a,b)) = 0,\, \varepsilon > 0 \\ 1/\varepsilon & 0 < \mathcal{H}^{d_1}(X'(a,b)) \le +\infty,\, \varepsilon > 0 \end{cases}$$

since $0 < \mathcal{H}^1(X'(a,b)) \le +\infty$ (i.e., $0 < \mathcal{H}^1(\mathbb{R}\cap(a,b)) = b-a \le +\infty$), hence $z(\varepsilon, X'(a,b)) = 1/\varepsilon$.

Part 2. $r(\varepsilon, X'(a,b))$

Suppose:

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R\subseteq Y, \mathcal{H}^{d_1}(f^{-1}[R]\cap X_j(a,b)) = \mathcal{H}^{d_1}(X_j(a,b))\}$$

since $X'(a,b)$ is countably infinite, there exists a sequence of discrete finite sets (i.e., $\{X_j(a,b)\}_{j\in\mathbb{N}}$ where for all $j\in\mathbb{N}$, $X_j(a,b) = X'(a,b)$), $\liminf_{j\to\infty} X_j(a,b) = \limsup_{j\to\infty} X_j(a,b) = X'(a,b)$ (i.e., the set-theoretic limit[36]) and for all $j\in\mathbb{N}$, $0 < \mathcal{H}^1(X_j(a,b)) < +\infty$

$$\mathbf{R}(X_j(a,b)) = \inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) : R\subseteq Y, \mathcal{H}^{1}(f^{-1}[R]\cap X_j(a,b)) = \mathcal{H}^{1}(X_j(a,b))\}$$

Hence, $0 \le \dim_{\mathrm{H}}(\mathrm{Range}(f)) = 1$, since the range of the Conway Base-13 function is $\mathbb{R}$. In addition, $d_1 = 1$. Since $0 < \mathcal{H}^1(X_j(a,b)) = b-a < +\infty$ and $\mathcal{H}^1(f^{-1}[R]\cap X_j(a,b)) = \mathcal{H}^1(X_j(a,b))$, when $R_1 \supseteq X_j(a,b)$ is an arbitrary set, we want $f[R_1] = \{0\}$ since $f(x) = 0$ for "almost all" $x\in\mathbb{R}$ and $\mathcal{H}^1(R_1) = \mathcal{H}^1(\mathbb{R})$. Thus, the smallest $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R) = \mathcal{H}^1(R)$ can be is $\mathcal{H}^1(\{0\}) = 0$ which means $\mathbf{R}(X_j(a,b)) = 0$.

Therefore, $\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = |0-(b-a)| = b-a$, for all $j\in\mathbb{N}$ and $0 \le \lim_{j\to\infty}|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))| = b-a < +\infty$. Hence, using:

$$r(\varepsilon, X'(a,b)) = \begin{cases} \varepsilon & 0 \le \limsup\limits_{j\to\infty}\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| < +\infty,\, \varepsilon > 0 \\ 1/\varepsilon & \limsup\limits_{j\to\infty}\left|\mathbf{R}(X_j(a,b)) - \mathcal{H}^{d_1}(X_j(a,b))\right| = +\infty,\, \varepsilon > 0 \end{cases}$$

we have $r(\varepsilon, X'(a,b)) = 1/\varepsilon$.

Part 3. $h(\varepsilon, X'(a,b))$

Suppose $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.
Notice that $G(X'(a,b))$ is dense in $[a,b]\times\mathbb{R}$ and $f(x) = 0$ for almost all $x\in\mathbb{R}$. Hence, since $\dim_{\mathrm{H}}(G(X'(a,b))) = d'(G(X'(a,b))) = 1$ and "almost all" of $G(X'(a,b))$ is a horizontal line segment with arc length $b-a$:

$$\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = \mathcal{H}^1(G(X'(a,b))) = b-a$$

In addition, suppose $D_1 \subseteq (a,b)$ is an arbitrary set such that for all $x\in D_1$, $f(x) = 0$ and $D_2 = G(X'(a,b)) \setminus \{(x,0) : x\in D_1\}$. Since $\mathcal{H}^1(D_1) = b-a$ and $\mathcal{H}^1(D_2) = 0$ are finite, there exists a sequence of sets $\{\mathsf{G}_j(X'(a,b))\}_{j\in\mathbb{N}}$, where $\inf_{j\to\infty}\mathsf{G}_j(X'(a,b)) = \inf_{j\to\infty}\mathsf{G}_j(X'(a,b)) = G(X'(a,b))$ and $0 < \mathcal{H}^1(\mathsf{G}_j(X'(a,b))) < +\infty$ for all $j\in\mathbb{N}$

Therefore, since $(\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty))$ and $\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0$, using:

$$h(\varepsilon, X'(a,b)) = \begin{cases} 1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) = 0,\, \varepsilon > 0 \\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b))) \in (0,+\infty)), \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b))) \neq 0 \end{cases}$$

we have $h(\varepsilon, X'(a,b)) = 1$

[36] See Section 4.5, page 33

Part 4. $\mathtt{C}_0(d_1, \ell, X'(a,b))$

Suppose, $d_1 = 0$ and $\#|\cdot|$ is the counting measure.
Moreover, $X'(a,b) = (a,b)$ and $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.

Thus, note:

$$\mathcal{P}_k(d_1, G(X'(a,b))) = \left\{\mathbf{G} \subseteq G(X'(a,b)) : \mathcal{H}^{d_1}(\mathbf{G}) = k\right\}$$
$$G_k(d_1, X'(a,b)) \in \mathcal{P}_k(d_1, G(X'(a,b)))$$

where $\mathbb{G}_0(d_1, X'(a,b))$ is the set of all limit points of $G(X'(a,b)) \setminus G_0(d_1, X'(a,b))$. Since $\dim_{\mathrm{H}}(G(X'(a,b))) = 1$, there exists $\mathbf{G}$ such that $\mathcal{H}^1(G) = 0$: when $D_1 \subseteq (a,b)$ is an arbitrary set where for all $x \in D_1$, $f(x) = 0$ and $D_2 = G(X'(a,b)) \setminus \{(x,0) : x \in D_1\}$, then note $\mathcal{H}^1(D_2) = 0$. Therefore, $G_0(d_1, X'(a,b))$ is arbitrary. However, since D_2 is a dense subset of $G(X'(a,b))$ and $G(X'(a,b)) \setminus D_2 = \{(x,0) : x \in D_1\}$, so $\mathbb{G}_0(d_1, X'(a,b)) = \{(x,0) : x \in D_1\}$ and intersects any vertical line once with an x-intercept in $\mathbb{R}$.

Moreover, $\ell(X'(a,b)) \subset \mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$ and since $X'(a,b)$ is non-empty, $\ell(X'(a,b))$ can exist.
Hence, whenever:

$$\mathbf{C}(X) = \begin{cases} 1/\varepsilon & \#|X| = 0, \varepsilon > 0 \\ \#|X| & 0 < \#|X| < +\infty \\ \varepsilon & \#|X| = +\infty,\, \varepsilon > 0 \end{cases} \tag{48}$$

since

$$\mathtt{C}_0(d_1, \ell, X'(a,b)) = \inf_{G_0(d_1, X'(a,b)) \in \mathcal{P}_0(d_1, X'(a,b))} \mathbf{C}(\mathbb{G}_0(d_1, X'(a,b)) \cap \ell(X'(a,b))$$
$$\inf_{G_0(d_1, X'(a,b)) \in \mathcal{P}_0(d_1, X'(a,b))} \mathbf{C}(\mathbb{G}(X'(a,b)) \cap \ell(X'(a,b)))$$

w.r.t. the counting measure $\#|\cdot|$ and Equation 48, $\#|\mathbb{G}(d_1, X'(a,b)) \cap \ell(X'(a,b))| = 1$ and $\mathbf{C}(\mathbb{G}(d_1, X'(a,b)) \cap \ell(X'(a,b))) = 1$

Part 5. $\lim_{\varepsilon\to 0} \mathbf{M} = \lim_{\varepsilon \to 0}(z \cdot r \cdot h \cdot \mathtt{C}_0)$

In Case 10 of Section 3.3.10 Part 1, 2, 3, and 4, where:

- $z(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $r(\varepsilon, X'(a,b)) = \varepsilon$
- $h(\varepsilon, X'(a,b)) = 1$
- $\mathtt{C}_0(d_1, \ell, X'(a,b)) = 1$

$$\lim_{\varepsilon\to 0} \mathbf{M}(\varepsilon, d_1, \ell, X'(a,b)) = \tag{49}$$
$$\lim_{\varepsilon\to 0} \left(z(\varepsilon, X'(a,b)) \cdot r(\varepsilon, X'(a,b)) \cdot h(\varepsilon, X'(a,b)) \cdot \mathtt{C}_0(d_1, \ell, X'(a,b))\right) = \lim_{\varepsilon \to 0} (1/\varepsilon \cdot \varepsilon \cdot 1 \cdot \varepsilon) = +\infty \tag{50}$$

Part 6. **Applying The Measure of Discontinuity to** f

For every $B \subseteq X'(a,b)$, $\mathfrak{c} = 1$ (see Equation 49):

$$\lim_{\varepsilon\to 0} \mathbf{M}(\varepsilon, d_1, \ell, B) = \lim_{\varepsilon \to 0}(1/\varepsilon \cdot \varepsilon \cdot 1 \cdot 1) = 1 = \mathfrak{c} \tag{51}$$

Thus, when $\mathfrak{c} = 0$ and $\mathfrak{c} > 1$, B is the empty set. Hence, when $\mathfrak{c} = 1$, $B = X'(a,b)$.

Now, suppose $\liminf_{j\to\infty} \mathbf{X}_j(a,b) = \limsup_{j\to\infty} \mathbf{X}_j(a,b) = X'(a,b)$ (i.e., the set theoretic limit[37]) where for all $j \in \mathbb{N}$, $0 < \mathcal{H}^1(\mathbf{X}_j(a,b)) < +\infty$. Thus, for all $j \in \mathbb{N}$, $\mathbf{X}_j(a,b) = X'(a,b)$ since $0 < \mathcal{H}^1(X'(a,b)) = b - a < +\infty$.
Hence:

$$\mathcal{M}_j(a, b, d_1, t) = \tag{52}$$

[37]See Section 4.5, page 33

$$\left(\frac{\sum_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq X'(a,b),\ \lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b))\}}\right)=$$

$$\frac{\max\{0,\min\{0-1,t\}\}\cdot\sup\left(\mathcal{H}^1\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq\emptyset,\ \lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=0\right\}\right)\right)}{\min\{1/t,\mathcal{H}^1(X'(a,b))\}}+$$

$$\frac{\max\{0,\min\{1-1,t\}\}\cdot\sup\left(\mathcal{H}^1\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq\emptyset,\ \lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=1\right\}\right)\right)}{\min\{1/t,\mathcal{H}^1(X'(a,b))\}}+$$

$$\frac{\sum_{\mathfrak{c}\in\{2,3,\cdots\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^1\left(\left\{B\cap\mathbf{X}_j(a,b):B\subseteq\emptyset,\ \lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^1(X'(a,b))\}}=$$

$$\frac{\max\{0,\min\{-1,t\}\}\cdot\mathcal{H}^1(\emptyset)}{\max\{1/t,\mathcal{H}^1(B\cap\mathbf{X}_j(a,b))\}}+\frac{\max\{0,\min\{0,t\}\}\cdot\mathcal{H}^1(B\cap\mathbf{X}_j(a,b))}{\max\{1/t,\mathcal{H}^1(B\cap\mathbf{X}_j(a,b))\}}+\frac{\sum_{\mathfrak{c}\in\{2,3,\cdots\}\cup\{\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^1(\emptyset)}{\max\{1/t,\mathcal{H}^1(B\cap\mathbf{X}_j(a,b))\}}=$$

$$\frac{\max\{0,-1\}\cdot0}{\max\{1/t,\mathcal{H}^1(B\cap X'(a,b))\}}+\frac{\max\{0,0\}\cdot\mathcal{H}^1(B\cap X'(a,b))}{\max\{1/t,\mathcal{H}^1(B\cap X'(a,b))\}}+\frac{\sum_{\mathfrak{c}\in\{2,3,\cdots\}\cup\{\infty\}}\max\{0,\mathfrak{c}-1\}\cdot0}{\max\{1/t,\mathcal{H}^1(B\cap X'(a,b))\}}=$$

$$\frac{-1\cdot0}{b-a}+\frac{0\cdot b-a}{b-a}+\frac{\sum_{\mathfrak{c}\in\{2,3,\cdots\}\cup\{\infty\}}0\cdot0}{b-a}=0$$

Hence, the measure of discontinuity $\mathcal{D}_{d_1}$ is:

$$\mathcal{D}_0=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\mathcal{M}_j(a,b,d_1,t)\right)\right)=\lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\mathcal{M}_j(a,b,d_1,t)\right)\right)=\lim_{(a,b)\to\infty}(0)=0$$

This verifies the blockquote at the top of Case 10, which states the measure of discontinuity $D_0(f)$ should be zero.

3.3.11. *Case 11.* Suppose $f : X \to Y$ is a function, $X = \mathbb{R}$, $Y = \mathbb{R}$, $X_1 = \mathbb{R}$, $d_1 = \dim_{\mathrm{H}}(X_1) = 1$, $\dim_{\mathrm{H}}(X) = 0$. Hence, $X' = X \cap X_1 = \mathbb{R}$.

We define f such that the graph of f is everywhere surjective[38] with zero Hausdorff measure in its dimension.

Note, this function can exist [5].

> In addition, Case 11 is an example of Section 2.2.2, criteria 8, so the measure of discontinuity $\mathcal{D}_{d_1}(f)$ needs to be positive infinity.
> See Section 4.4, page 33, for an explicit example.

For parts 1-5, we compute the components $z(\varepsilon, X'(a,b))$, $r(\varepsilon, X'(a,b))$, $h(\varepsilon, X'(a,b))$, $\mathtt{C}_0(d_1,\ell,X'(a,b))$ and $\lim_{\varepsilon\to0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b))$ (i.e., their motivation is in Section 3.1, page 4), then use these components to compute the measure of discontinuity $\mathcal{D}_{d_1}(f)$ (part 6).

Part 1. $z(\varepsilon, X'(a,b))$

Suppose:

$$z(\varepsilon,X'(a,b))=\begin{cases}\varepsilon & \mathcal{H}^{d_1}(X'(a,b))=0,\ \varepsilon>0\\ 1/\varepsilon & 0<\mathcal{H}^{d_1}(X'(a,b))\le+\infty,\ \varepsilon>0\end{cases}$$

since $0 < \mathcal{H}^1(X'(a,b)) \le +\infty$ (i.e., $\mathcal{H}^1(\mathbb{Q}\cap(a,b)) = +\infty$), hence $z(\varepsilon, X'(a,b)) = 1/\varepsilon$.

Part 2. $r(\varepsilon, X'(a,b))$

Suppose:

$$\mathbf{R}(X_j(a,b))=\inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R):R\subseteq Y,\mathcal{H}^{d_1}(f^{-1}[R]\cap X_j(a,b))=\mathcal{H}^{d_1}(X_j(a,b))\}$$

[38] See 4.4, page 33

since $X'(a,b)$ is countably infinite, there exists a sequence of discrete finite sets (i.e., $\{X_j(a,b)\}_{j\in\mathbb{N}}$ where for all $j\in\mathbb{N}$, $X_j(a,b)=X'(a,b)$), $\liminf_{j\to\infty}X_j(a,b)=\limsup_{j\to\infty}X_j(a,b)=X'(a,b)$ (i.e., the set-theoretic limit[39]) and for all $j\in\mathbb{N}$, $0<\mathcal{H}^1(X_j(a,b))<+\infty$

$$\mathbf{R}(X_j(a,b))=\inf\{\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R):R\subseteq Y,\mathcal{H}^{d_1}(f^{-1}[R]\cap X_j(a,b))=\#|X_j(a,b)|\}$$

Hence, $\dim_{\mathrm{H}}(\mathrm{Range}(f))>1$, since f is everywhere surjective such that its graph has zero Hausdorff measure in its dimension. In addition, $d_1=1$. Since $0<\mathcal{H}^1(X_j(a,b))<+\infty$ and $\mathcal{H}^1(f^{-1}[R]\cap X_j(a,b))=\mathcal{H}^1(X_j(a,b))$, then when $R_1\supseteq X_j(a,b)=(a,b)$ is an arbitrary set, $f[R_1]=\mathbb{R}$ for any $R_1\subseteq X_j(a,b)$ since f is everywhere surjective and its graph has zero Hausdorff measure in its dimension. Thus, the smallest $\mathcal{H}^{\dim_{\mathrm{H}}(\mathrm{Range}(f))}(R)=\mathcal{H}^1(R)$ can be is $+\infty$. That means $\mathbf{R}(X_j(a,b))=+\infty$.

Therefore, $\left|\mathbf{R}(X_j(a,b))-\mathcal{H}^{d_1}(X_j(a,b))\right|=|(+\infty)-(b-a)|=0$, for all $j\in\mathbb{N}$ and $\lim_{j\to\infty}|\mathbf{R}(X_j(a,b))-\mathcal{H}^{d_1}(X_j(a,b))|=+\infty$. Thus, using:

$$r(\varepsilon,X'(a,b))=\begin{cases}\varepsilon & 0\le\limsup\limits_{j\to\infty}\left|\mathbf{R}(X_j(a,b))-\mathcal{H}^{d_1}(X_j(a,b))\right|<+\infty,\ \varepsilon>0\\ 1/\varepsilon & \limsup\limits_{j\to\infty}\left|\mathbf{R}(X_j(a,b))-\mathcal{H}^{d_1}(X_j(a,b))\right|=+\infty,\ \varepsilon>0\end{cases}$$

we have $r(\varepsilon,X'(a,b))=1/\varepsilon$.

Part 3. $h(\varepsilon,X'(a,b))$

Suppose $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.
Notice that $\dim_{\mathrm{H}}(G(X'(a,b)))>1$, f is everywhere surjective and the graph of f has zero Hausdorff measure in its dimension. Hence:

$$\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b)))=\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b)))=0$$

Therefore, using:

$$h(\varepsilon,X'(a,b))=\begin{cases}1/\varepsilon & \mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b)))=0,\ \varepsilon>0\\ 1 & (\forall j)(\mathcal{H}^{d_1}(\mathsf{G}_j(X'(a,b)))\in(0,+\infty)),\mathcal{H}^{d'(G(X'(a,b)))}(G(X'(a,b)))\neq 0\end{cases}$$

we have $h(\varepsilon,X'(a,b))=1/\varepsilon$

Part 4. $\mathtt{C}_0(d_1,\ell,X'(a,b))$

Suppose, $d_1=0$ and $\#|\cdot|$ is the counting measure.
Moreover, $X'(a,b)=(a,b)$ and $G(X'(a,b))$ is the graph of $f|_{X'(a,b)}$.

Thus, note:

$$\mathcal{P}_k(d_1,G(X'(a,b)))=\left\{\mathbf{G}\subseteq G(X'(a,b)):\mathcal{H}^{d_1}(\mathbf{G})=k\right\}$$
$$G_k(d_1,X'(a,b))\in\mathcal{P}_k(d_1,G(X'(a,b)))$$

where $\mathbb{G}_0(d_1,X'(a,b))$ is the set of all limit points of $G(X'(a,b))\setminus G_0(d_1,X'(a,b))$. Since $\dim_{\mathrm{H}}(G(X'(a,b)))=1$, there exists $\mathbf{G}$ such that $\mathcal{H}^1(G)=0$. However, because $\dim_{\mathrm{H}}(G(X'(a,b)))=d'(G(X'(a,b)))>1$, therefore $\mathcal{H}^1(G(X'(a,b)))=+\infty$. Hence, $G_0(d_1,X'(a,b))$ is the empty set. Also, since $X'(a,b)=(a,b)$, $\mathbb{G}_0(d_1,X'(a,b))=[a,b]\times\mathbb{R}$ and intersects any vertical line with an x-intercept in $\mathbb{R}$.

Moreover, $\ell(X'(a,b))\subset\mathbb{R}^2$ is an arbitrary, vertical line whose x-intercept is an element of $X'(a,b)$ and since $X'(a,b)$ is non-empty, $\ell(X'(a,b))$ can exist.
Hence, whenever:

$$\mathbf{C}(X)=\begin{cases}1/\varepsilon & \#|X|=0,\varepsilon>0\\ \#|X| & 0<\#|X|<+\infty\\ \varepsilon & \#|X|=+\infty,\ \varepsilon>0\end{cases}\tag{53}$$

since

$$\mathtt{C}_0(d_1,\ell,X'(a,b))=\inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))}\mathbf{C}(\mathbb{G}_0(d_1,X'(a,b))\cap\ell(X'(a,b))$$

[39] See Section 4.5, page 33

$$\inf_{G_0(d_1,X'(a,b))\in\mathcal{P}_0(d_1,X'(a,b))} \mathbf{C}(\mathbb{G}(X'(a,b))\cap\ell(X'(a,b)))$$

w.r.t. the counting measure $\#|\cdot|$ and Equation 53, $\#|\mathbb{G}(d_1,X'(a,b))\cap\ell(X'(a,b))| = +\infty$ and $\mathbf{C}(\mathbb{G}(d_1,X'(a,b))\cap \ell(X'(a,b))) = \varepsilon$

Part 5. $\lim_{\varepsilon\to 0}\mathbf{M} = \lim_{\varepsilon\to 0}(z\cdot r\cdot h\cdot \mathtt{C}_0)$

In Case 11 of Section 3.3.11 Part 1, 2, 3, and 4, where:

- $z(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $r(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $h(\varepsilon, X'(a,b)) = 1/\varepsilon$
- $\mathtt{C}_0(d_1,\ell,X'(a,b)) = \varepsilon$

$$\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,X'(a,b)) = \tag{54}$$

$$\lim_{\varepsilon\to 0}\left(z(\varepsilon,X'(a,b))\cdot r(\varepsilon,X'(a,b))\cdot h(\varepsilon,X'(a,b))\cdot \mathtt{C}_0(d_1,\ell,X'(a,b))\right) = \lim_{\varepsilon\to 0}\left(1/\varepsilon\cdot\varepsilon\cdot 1\cdot\varepsilon\right) = +\infty \tag{55}$$

Part 6. **Applying The Measure of Discontinuity to** f

Since, for every $B\subseteq X'(a,b)$, $\mathfrak{c} = +\infty$ (see Equation 54):

$$\lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B) = +\infty = \mathfrak{c}$$

Hence, when $\mathfrak{c} = +\infty$ and $B = \mathbf{X}_j(a,b)$, such that:

$$\limsup_{j\to\infty}\mathbf{X}_j(a,b) = \liminf_{j\to\infty}\mathbf{X}_j(a,b) = X'(a,b)$$

and:

$$0 < \mathcal{H}^{\dim_{\mathrm{H}}(X')}(\mathbf{X}_j(a,b)) < +\infty$$

$\mathcal{H}^{\dim_{\mathrm{H}}(X')}(B) = \mathcal{H}^1(B) = b-a$ is the largest possible value. In addition, when $0\le\mathfrak{c}<+\infty$, $\mathcal{H}^1(B) = \mathcal{H}^1(\emptyset) = 0$. Therefore, since $X_j(a,b) = X'(a,b)$, for all $j\in\mathbb{N}$:

$$\begin{aligned}
&\mathcal{M}(a,b,d_1,t) = \qquad (56)\\
&\left(\frac{\sum_{\mathfrak{c}\in\mathbb{N}\cup\{0\}\cup\{+\infty\}}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^{\dim_{\mathrm{H}}(X')}\left(\left\{B\cap\mathbf{X}_j(a,b) : B\subseteq X'(a,b), \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^{\dim_H(X')}(\mathbf{X}_j(a,b))\}}\right) = \\
&\frac{\sum_{\mathfrak{c}=0}^{\infty}\max\{\mathfrak{c},\min\{\mathfrak{c}-1,t\}\}\cdot\sup\left(\mathcal{H}^1\left(\left\{B\cap\mathbf{X}_j(a,b) : B\subseteq X'(a,b), \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=\mathfrak{c}\right\}\right)\right)}{\min\{1/t,\mathcal{H}^1(\mathbf{X}_j(a,b))\}} + \\
&\frac{\max\{0,\min\{\infty-1,t\}\}\cdot\sup\left(\mathcal{H}^1\left(\left\{B\cap\mathbf{X}_j(a,b) : B\subseteq X'(a,b) : \lim_{\varepsilon\to 0}\mathbf{M}(\varepsilon,d_1,\ell,B)=+\infty\right\}\right)\right)}{\min\{1/t,\mathcal{H}^1(\mathbf{X}_j(a,b))\}} = \\
&\frac{\sum_{\mathfrak{c}=1}^{\infty}\max\{0,\min\{\mathfrak{c}-1,t\}\}\cdot\mathcal{H}^1(\emptyset)}{\max\{1/t,\mathcal{H}^1(X'(a,b)\}} + \frac{\max\{0,\min\{+\infty,t\}\}\cdot\mathcal{H}^1(X_j(a,b))+\mathbf{X}_j(a,b)|}{\max\{1/t,\mathcal{H}^1(X'(a,b))\}} = \\
&\frac{\sum_{\mathfrak{c}=1}^{\infty}\max\{0,\mathfrak{c}-1\}\cdot 0}{\max\{1/t,b-a\}} + \frac{\max\{0,t\}\cdot b-a}{\max\{1/t,b-a\}} = \\
&\frac{0}{b-a} + \frac{t\cdot b-a}{b-a} = t
\end{aligned}$$

Thus, the measure of discontinuity $\mathcal{D}_{d_1}$ is:

$$\mathcal{D}_1 = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\limsup_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}\left(\liminf_{j\to\infty}\mathcal{M}(a,b,d_1,t)\right)\right) = \lim_{(a,b)\to\infty}\left(\lim_{t\to\infty}t\right) = +\infty$$

This verifies the blockquote at the top of Case 11, which states the measure of discontinuity $\mathcal{D}_1(f)$ should be positive infinity.

4. Appendix

4.1. **Definition of Continuity.** Suppose $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ are arbitrary sets. The continuity of $f : X \to Y$ at $x_0 \in X$ means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X \setminus \{x_0\}$

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \epsilon$$

Hence, f is continuous on arbitrary set $X_1 \subseteq \mathbb{R}$, where:

(1) $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension
(2) $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimensions on the Borel σ-algebra

whenever:

$$C_{\mathcal{M}}(f, X_1) = \mathcal{H}^{\dim_{\mathrm{H}}(X_1)}(X_1 \setminus X) = 0 \tag{57}$$

4.2. **"Splitting" the Closure of the Graph of $f : X \to Y$ into c Functions Continuous on the Set X.** Let $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ be arbitrary sets. Suppose $\mathrm{C_g} = \mathbf{cl}(\text{graph}(f))$ is the topological closure of the graph of function $f : X \to Y$. Hence, for all $i \in \{1, 2, \cdots, c\}$, $\{(x, f_i(x)) : x \in \text{dom}(f_i)\} \subseteq \mathrm{C_g}$ such that $C_{\mathcal{M}}(f_i, X) = 0$.

4.2.1. *Example.* Consider the Dirichlet Function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Note, $\mathrm{C_g} = \{(x, 0), (x, 1) : x \in \mathbb{R}\}$. Thus, $i \in \{1, 2\}$, $\{(x, i - 1) : x \in \mathbb{R}\} \subseteq \mathrm{C_g}$ such that $C_{\mathcal{M}}(f_i, X) = 0$ (in Section 4.1, for all $i \in \{1, 2\}$, $\text{dom}(f_i) = \mathbb{R}$, $X = \mathbb{R}$, and $f_i : \mathbb{R} \to \mathbb{R}$ is continuous such that $C_{\mathcal{M}}(f_i, X) = \mathcal{H}^{\dim_{\mathrm{H}}(X)}(X \setminus \text{dom}(f_i)) = \mathcal{H}^1(\mathbb{R} \setminus \mathbb{R}) = \mathcal{H}^1(\emptyset) = 0$).

4.3. **Hyper-discontinuous function.** Suppose $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$.

Definition: A function $f : X \to Y$ is *hyper-discontinuous* if for every $x \in X$, $\exists\, \delta > 0$, $\varepsilon > 0$ such that $y \in X \setminus \{x\}$, $|y - x| < \delta$, $\implies |f(x) - f(y)| \geq \varepsilon$.

4.3.1. *Explicit Example of a Hyper-Discontinuous Functions.* Consider the function $f : \mathbb{Q} \cap [0, 1] \to \mathbb{R}$, where $f(p/q) = 1/q$ for all coprime integers $p, q \in \mathbb{Z}$

4.4. **Everywhere Surjective Function.** Let (X, T) be a standard topology. A function $f : X \to Y$ is *everywhere surjective* from X to Y, if $f[X] = Y$ for every $X \in \mathrm{T}$. (See [2], for more info).

4.4.1. *Explicit Example of a Everywhere Surjective Function.* Consider an everywhere surjective $f : \mathbb{R} \to \mathbb{R}$ whose graph has zero Hausdorff measure in its dimension.

Since $(\mathbb{R}, \mathrm{T})$ is the standard topology, hence $f[(a, b)] = \mathbb{R}$ for every non-empty open interval (a, b). See this citation [5] for an explicit example.

4.5. **Set Theoretic Limit.** Suppose $(X_j)_{j \in \mathbb{N}}$ is a sequence of sets. The set-theoretic limit of a sequence of sets $(X_j)_{j \in \mathbb{N}}$ is X, whenever:

$$\limsup_{j \to \infty} X_j = \bigcap_{j \geq 1} \bigcup_{q \geq j} X_q$$

$$\liminf_{r \to \infty} X_j = \bigcup_{j \geq 1} \bigcap_{q \geq j} X_q$$

where:

$$\limsup_{j \to \infty} X_j = \liminf_{j \to \infty} X_j = X \tag{58}$$

4.6. **Explicit Example of Cases 4 and 6.** Suppose, $X = \mathbb{R}$, $Y = \mathbb{R}$, $X_1 = \mathbb{Q}$ (Case 4), $X_1 = \mathbb{R}$ (Case 6), $\dim_{\mathrm{H}}(X) = 1$, and the closure of the graph of f can be split into $\mathbf{c_1}$ functions continuous[40] on X_1. (Notice, $d_1 < \dim_{\mathrm{H}}(X)$).

[40] Section 4.2, page 33, where $X = X_1$

4.6.1. *Explicit Example.* Consider the following [6]:

- Partition $\mathbb{R}$ into sets A and B, such that A and B have a positive $\mathcal{H}^1$ measure in every non-empty open interval (a,b) and $\lim_{c\to\infty}\mathcal{H}^1(A\cap(-c,c))/(2c)\neq\lim_{c\to\infty}\mathcal{H}^1(B\cap(-c,c))/(2c)$.
  For simplicity, partition the unit interval $[0,1)$, then repeat that partition so $x\in A$ when $x-\lfloor x\rfloor\in A$.
  Let $A_1=[0,2/3)$. Let $B_1=(2/3,1]$.
  If $\bigcup_{n=1}^{\infty}A_n=A$, then for each interval in A_n remove the middle $1/2^{n+1}$ of the interval and send it to B_n. Similarly, for each interval in B_n, remove the middle $1/2^n$ of the interval and send it to A_n. Each set less than what is removed plus what is transferred determines A_{n+1},B_{n+1}. Thereby, $\bigcup_{n=1}^{\infty}B_n=B$.
- $\mathrm{N}_1=\bigcup_{n_1\in\mathbb{Z}}[\frac{4n_1}{3},\frac{4n_1+2}{3}]$
- $\mathrm{N}_2=\bigcup_{n_2\in\mathbb{Z}}[\frac{4n_2+1}{3},\frac{4n_2+2}{3}]$
- $\mathrm{N}_3=\bigcup_{n_3\in\mathbb{Z}}[\frac{4n_3+2}{3},\frac{4n_3+4}{3}]$
- $\mathrm{N}_2\subseteq\mathrm{N}_1$
- $\mathrm{N}_1\cap\mathrm{N}_3=\emptyset$
- $\mathrm{N}_2\cap\mathrm{N}_3=\emptyset$
- $Q_1=\{m_1/n_1:m_1\in\text{odd }\mathbb{N},n_1\in\text{even }\mathbb{N}\}$
- $Q_2=\{m_2/n_2:m_2,n_2\in\text{odd }\mathbb{N}\}$
- $Q_3=\{m_3/n_3:m_3\in\text{even }\mathbb{N},n_3\in\text{odd }\mathbb{N}\}$
- $Q_1\cap Q_2\cap Q_3=\emptyset$

The following is an explicit example of $f:\mathbb{R}\to\mathbb{R}$ matching Case 4 and Case 6:

$$f(x)=\begin{cases}x & x\in(A\setminus\mathbb{Q})\cap\mathrm{N}_1\\ x\sin(x) & x\in(B\setminus\mathbb{Q})\cap\mathrm{N}_2\\ x+\sin(x) & x\in Q_1\cup\mathrm{N}_3\\ x+\sin(x)+\cos(x) & x\in Q_2\\ x+\sin(x)\cos(x+1) & x\in Q_3\end{cases}\tag{59}$$

4.7. **Explicit Example of Case 5 and 7.** Suppose $f:X\to Y$ is a function, where $X=\mathbb{Q}$, $Y=\mathbb{R}$, $X_1=\mathbb{R}$ (Case 5), $X_1=\mathbb{Q}$ (Case 7), and $\dim_{\mathrm{H}}(X)=1$.

4.7.1. *Explicit Example.* Consider the following:

- $\mathrm{N}_1=\bigcup_{n_1\in\mathbb{Z}}[\frac{4n_1}{3},\frac{4n_1+2}{3}]$
- $\mathrm{N}_2=\bigcup_{n_2\in\mathbb{Z}}[\frac{4n_2+1}{3},\frac{4n_2+2}{3}]$
- $\mathrm{N}_3=\bigcup_{n_3\in\mathbb{Z}}[\frac{4n_3+2}{3},\frac{4n_3+4}{3}]$
- $\mathrm{N}_2\subseteq\mathrm{N}_1$
- $\mathrm{N}_1\cap\mathrm{N}_3=\emptyset$
- $\mathrm{N}_2\cap\mathrm{N}_3=\emptyset$
- $Q_1=\{m_1/n_1:m_1\in\text{odd }\mathbb{N},n_1\in\text{even }\mathbb{N}\}$
- $Q_2=\{m_2/n_2:m_2,n_2\in\text{odd }\mathbb{N}\}$
- $Q_3=\{m_3/n_3:m_3\in\text{even }\mathbb{N},n_3\in\text{odd }\mathbb{N}\}$
- $Q_1\cap Q_2\cap Q_3=\emptyset$

The following is an explicit example of $f:\mathbb{Q}\to\mathbb{R}$ matching Case 5 and Case 7:

$$f(x)=\begin{cases}x & x\in Q_1\cap N_1\\ x+\sin(x) & x\in Q_2\cap(N_2\cup N_3)\\ x+\sin(x)+\cos(2x) & x\in Q_3\cup((Q_1\cap N_3)\cup(Q_2\cap N_1))\end{cases}\tag{60}$$

4.8. **Proof The Example of Sets In The Family $\mathscr{X}=\{\mathsf{X}_r:r\in\{1,\cdots,\mathbf{c}_1\}\}$ On Case 5, page 15 are Pairwise Disjoint.** Suppose, we define the family of sets $\mathscr{X}=\{\mathsf{X}_r:r\in\{1,\cdots,\mathbf{c}_1\}\}$

$$\begin{cases}\mathsf{X}_r=\left\{s_1/\left(2^{(\mathbf{c_1}-1)}t_1\right):s_1,t_1\in\mathbb{Z}\right\} & r=1\\ \mathsf{X}_{\mathsf{r}}=\left\{s_r/\left(2^{(\mathbf{c_1}-r)}t_r\right):s_r,t_r\in\text{odd }\mathbb{Z}\right\}\setminus\{\mathsf{X}_1\} & 2\leq r\leq\mathbf{c_1}\end{cases}\tag{61}$$

4.8.1. *Proof The Sets in Family $\mathscr{X}$ Are Pairwise Disjoint.* Using Proof by Mathematical Induction:

(i) If $r=1$, then $\mathsf{X}_1=\left\{s_1/\left(2^{(\mathbf{c_1}-1)}t_1\right): s_1,t_1\in\mathbb{Z}\right\}$ is the only set defined in the family $\mathscr{X}$, so the family $\mathscr{X}$ of sets is pairwise disjoint.

(ii) Suppose, the family of sets $\mathscr{X}_{\mathbf{c_1}}=\{\mathsf{X}_\mathsf{r}: r\in\{1,\cdots,\mathbf{c_1}\}\}$, for all $r\in\{1,\cdots,\mathbf{c_1}\}$, are pairwise disjoint.

Hence, suppose $r=\mathbf{c_1}+1$, where $\mathscr{X}_{\mathbf{c_1}+1}=\mathscr{X}_{\mathbf{c_1}}\cup\{\mathsf{X}_{\mathbf{c_1}+1}\}$. Note,

$$\mathsf{X}_{\mathbf{c_1}+1}=\left\{s_{\mathbf{c_1}}/\left(2^{((\mathbf{c_1}+1)-\mathbf{c_1})}t_{\mathbf{c_1}}\right): s_{\mathbf{c_1}},t_{\mathbf{c_1}}\in\text{odd}\,\mathbb{Z}\right\}\setminus\{\mathsf{X}_1\} \tag{62}$$

$$=\left\{s_{\mathbf{c_1}}/\left(2^{(\mathbf{c_1}+1-\mathbf{c_1})}t_{\mathbf{c_1}}\right): s_{\mathbf{c_1}},t_{\mathbf{c_1}}\in\text{odd}\,\mathbb{Z}\right\}\setminus\{\mathsf{X}_1\} \tag{63}$$

$$=\left\{s_{\mathbf{c_1}}/\left(2^{((\mathbf{c_1}-\mathbf{c_1})+1)}t_{\mathbf{c_1}}\right): s_{\mathbf{c_1}},t_{\mathbf{c_1}}\in\text{odd}\,\mathbb{Z}\right\}\setminus\{\mathsf{X}_1\} \tag{64}$$

$$=\left\{s_{\mathbf{c_1}}/\left(2\left(2^{(\mathbf{c_1}-\mathbf{c_1})}\right)t_{\mathbf{c_1}}\right): s_{\mathbf{c_1}},t_{\mathbf{c_1}}\in\text{odd}\,\mathbb{Z}\right\}\setminus\{\mathsf{X}_1\} \tag{65}$$

$$=\{(s_{\mathbf{c_1}}/2)/(t_{\mathbf{c_1}}): s_{\mathbf{c_1}},t_{\mathbf{c_1}}\in\text{odd}\,\mathbb{Z}\}\setminus\{\mathsf{X}_1\} \tag{66}$$

Since, $s_{\mathbf{c_1}}/2$ for all odd integers $s_{\mathbf{c_1}}$ is disjoint from all odd integers $s_{r'}=(2s_{r'})/2$ such that $r'\le\mathbf{c_1}$, hence $\mathscr{X}_{\mathbf{c_1}+1}$ is pairwise disjoint.

4.9. **Euler's Totient Function.** The totient function counts the number of positive integers up to a given integer n which are coprime to n. This function be used to count the number of fractions with a denominator less than n with a comprime numerator and denominator.

4.10. **Equidistribution.** A sequence of sets $(t_q)_{q\in\mathbb{Z}}$ is equidistributed in $[\alpha,\beta]$, when for all sub-intervals $[\alpha',\beta']\subseteq[\alpha,\beta]$.

$$\lim_{q\to\infty}\frac{|t_q\cap[\alpha',\beta']|}{|t_q\cap[\alpha,\beta]|}=\frac{\beta'-\alpha'}{\beta-\alpha}$$

4.11. **Conway Base-**13 **Function.** Consider this definition of the Conway Base-13 function [7]:

(1) Expand $x\in(0,1)$ in base 13, using digits $\{0,1,\cdots,d,m,p\}$ where $d=10$. Note, for any rational number which is a fully simplified fraction a/b such that b is a power of 13, there exists two such expansions: a terminating expansion, and a non-terminating one ending in repeated p digits. In such a case, use the terminating expansion.

(2) Let $S\subseteq(0,1)$ be the set of reals that is an expansion involving finitely many p, m and d digits, such that the final d digit occurs after the final p and m digit. (We may require that there be at least one digit $0-9$ between the final p and m digit, but this does not seem necessary.) Then, every $x\in S$ has a base 13 expansion of the form

$$0.x_1x_2\ldots x_n\ [\ p \text{ or } m\ ]\ a_1a_2\ldots a_k\ [\ \text{d}\ ]\ b_1b_2\ldots$$

for some digit $x_j\in\{0,\ldots,p\}$ and where the digits a_j and b_j are limited to $\{0,\ldots,9\}$ for all j. The square brackets above are intended for emphasis; and in particular, the $n+1^{\text{st}}$ base-13 digits of x is the final occurance of either p or m in the expansion of x.

(3) For $x\in S$, we define $f(x)$ by transliterating the string format above. We ignore the digits x_1 through x_n, transliterate the p or m as a plus-sign and minus-sign, and d as a decimal point. This yeilds a decimal expansion for a real number, either

$$+a_1a_2\ldots a_k.b_1b_2\ldots$$

or

$$-a_1a_2\ldots a_k.b_1b_2\ldots$$

according to whether the $n+1^{\text{st}}$ base-13 digit of x is a p or an m respectively. For $x\in S$, we set $f(x)$ to this number; for $x\in S$, we set $f(x)=0$

Note: this function is not computable, as there is no way to determine (in advance) whether the base-13 expansion of $x \in (0,1)$ has only finitely many occurrences of the digits p, m, or d. Even if one is provided with a number which is promised to have only finitely many, in general one cannot know when they found the last one. Regardless, if one is provided with a number $x \in (0,1)$ for which they know the location of the final p, m, and d digits, they can compute $f(x)$ very straightforwardly.

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