DEFINING A SATISFYING EXPECTED VALUE FROM CHOSEN SEQUENCES OF BOUNDED FUNCTIONS CONVERGING TO PATHOLOGICAL FUNCTIONS (V2)

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Abstract. Let $n \in \mathbb{N}$ and suppose function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel. We want a satisfying average for all pathological f (e.g., a everywhere surjective f whose graph has zero Hausdorff measure in its dimension) taking finite values only. If this is impossible, we wish to average a nowhere continuous f defined on the rationals. The problem is that the expected value of these examples of f, w.r.t the Hausdorff measure in its dimension, is undefined. Thus, take any chosen sequence of bounded functions converging to f with the same satisfying and finite expected value.

Note, "satisfying" is explained in the leading question which uses rigorous versions of phrases in the former paragraph and the "measure" of the sequence of each bounded functions' graph which involves minimal pair-wise disjoint covers of the graph with equal ε measure, sample points from each cover, paths of line segments between sample points, the lengths of the line segments in the path, removed lengths which are outliers, remaining lengths which are converted into a probability distribution, and the entropy of the distribution. We also explain "satisfying" by defining the actual rate expansion of the sequence of each bounded functions' graph and the "rate of divergence" of the sequence compared to that of other sequences.

Keywords. Pathalogical Functions, Hausdorff measure, Expected Value, Function Space, Prevalent and Shy Sets, Covers, Samples, Euclidean Distance, Entropy, Choice Function

1. Intro

Let $n \in \mathbb{N}$ and suppose function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel. We want a satisfying average for all pathological f taking finite values only. The problem is that the expected value of specific examples of f (§2.1,§2.2), w.r.t the Hausdorff measure in its dimension, is undefined (§2.3). To fix this, we take the expected value of a sequence of bounded functions converging to f (§2.3.2); however, depending on the sequence of bounded functions chosen, the expected value could be one of several values (thm. 1). Hence, we define a leading question (§3.1) that chooses sequences of bounded functions with the same satisfying and finite expected value, such that the term "satisfying" is explained rigorously.

Note, the leading question (§3.1) was inspired by two problems (i.e., informal versions of thm. 2 and 5):

(1) If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where the expected value of f w.r.t the Hausdorff measure *in its dimension* is finite, then F is shy (§2.4).
   - If $F \subset \mathbb{R}^A$ is shy, we say "**almost no**" element of $\mathbb{R}^A$ lies in F

(2) If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where two sequences of bounded functions that converge to f have different expected values, then F is prevelant (§2.4)
   - If $F \subset \mathbb{R}^A$ is prevelant, we say "**almost all**" elements of $\mathbb{R}^A$ lies in F

In section §5, we clarify the leading question (§3.1) by applying the rigorous definitions of the leading question to specific examples (§5.2.1). We also define a "measure" (§5.3.1,§5.3.2) of the sequence of each bounded functions' graph. This is crucial for defining a satisfying expected value, where the "measure" is defined by the following:

(1) Covering each graph with minimal, pairwise disjoint sets of equal ε Hausdorff measure (§5.3.1, step 1)
(2) Taking a sample point from each cover (§5.3.1, step 2)
(3) Taking a "pathway of line segments" starting with sample point x_0 to the sample point with the smallest Euclidean distance from x_0 (i.e., when more than one point has the smallest Euclidean distance to x_0, take either of those points). Next, repeat this process until the pathway intersects with every sample point once (§5.3.1, step 3a)

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(4) Taking the length of each line segments in the pathway and remove the outliers which are more than $C > 0$ times the interquartile range of the length of each line segment as $\varepsilon \to 0$ (§5.3.1, step 3b)
(5) Multiply the remaining lengths by a constant to get a probability distribution (§5.3.1, step 3c)
(6) Taking the entropy of the distribution (§5.3.1, step 3d)
(7) Taking the maximum entropy w.r.t all pathways (§5.3.1, step 3e)

We give examples of how to apply the "measure" (§5.3.3-§5.3.5), then define the actual rate of expansion of the sequence of each bounded functions' graph (§5.4).

Finally, we answer the leading question in §6. Since the answer is complicated, is likely incorrect, and the leading question might not admit an unique expected value, it is best to keep refining the leading question (§3.1) rather than worrying about an immediate solution.

2. FORMALIZING THE INTRO

Let $n \in \mathbb{N}$ and suppose function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel. Let $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension, where $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra.

We want an unique, satisfying average for each of the following functions (§2.1,§2.2) taking finite values only. We explain the method of averaging in later sections, starting from §2.3.1.

2.1. First special case of f.

If the graph of f is G, we want an explicit f where:

(1) The function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is everywhere surjective

> Let (A, T) be a standard topology. A function $f : A \to \mathbb{R}$ is *everywhere surjective* from A to $\mathbb{R}$, if $f[V] = \mathbb{R}$ for every $V \in \mathrm{T}$.

(2) $\mathcal{H}^{\dim_{\mathrm{H}}(G)}(G) = 0$

2.1.1. *Potential Example.*

If $A = \mathbb{R}$, using this post [3]:

> Consider a Cantor set $C \subseteq [0, 1]$ with Hausdorff dimension 0 [4]. Now consider a countable disjoint union $\cup_{m\in\mathbb{N}} C_m$ such that each C_m is the image of C by some affine map and every open set $O \subseteq [0, 1]$ contains C_m for some m. Such a countable collection can be obtained by e.g. at letting C_m be contained in the biggest connected component of $[0, 1] \setminus (C_1 \cup \cdots \cup C_{m-1})$ (with the center of C_m being the middle point of the component).
>
> Note that $\cup_m C_m$ has Hausdorff dimension 0, so $(\cup_m C_m) \times [0, 1] \subseteq \mathbb{R}^2$ has Hausdorff dimension one [2].
>
> Now, let $g : [0, 1] \to \mathbb{R}$ such that $g|_{C_m}$ is a bijection $C_m \to \mathbb{R}$ for all m (all of them can be constructed from a single bijection $C \to \mathbb{R}$, which can be obtained without choice, although it may be ugly to define) and outside $\cup_m C_m$ let g be defined by $g(x) = h(x)$, where $h : [0, 1] \to \mathbb{R}$ has a graph with Hausdorff dimension 2 [13] (this doesn't require choice either).
>
> Then the function g has a graph with Hausdorff dimension 2 and is everywhere surjective, but its graph has Lebesgue measure 0 because it is a graph (so it admits uncountably many disjoint vertical translates).
>
> Note, we can make the construction with union of C_m rather explicit as follows. Split the binary expansion of x as strings of size with a power of two, say $x = 0.1101000010\ldots$ becomes $(s_0, s_1, s_2, \ldots) = (1, 10, 1000, \ldots)$. If this sequence eventually contains only strings of the form $0 \cdots 0$ or $1 \cdots 1$, say after s_k, then send it to $y = \sum_{i>0} \epsilon_i 2^{-i}$, where $s_{k+i} = \epsilon_i \cdots \epsilon_i$. Otherwise, send it to the explicit continuous function h given by the linked article [13]. This will give you something from $[0, 1) \to [0, 1)$
>
> Finally, compose an explicit (reasonable) bijection from $[0, 1)$ to $\mathbb{R}$. In your case, the construction can be easily adapted so that the $[0, 1]$ or $[0, 1)$ target space is actually $(0, 1)$, then compose with $t \mapsto (1 - 2x)/(x^2 - x)$.

In case we cannot obtain a unique, satisfying average (§3.1) from §2.1.1, consider the following:

2.2. Second special case of f.

Suppose, we define $A = \mathbb{Q}$, where $f : A \to \mathbb{R}$, such that:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \end{cases} \tag{1}$$

In the next section, we state why we want §2.1 and §2.2.

2.3. **Attempting to Analyze/Average f.** Suppose, the expected value of f w.r.t the Hausdorff measure in its dimension is:

$$\mathbb{E}[f]=\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A)}(A)}\int_A f\,d\mathcal{H}^{\dim_{\mathrm{H}}(A)} \tag{2}$$

Then, using §2.1.1, the integral of f w.r.t the Hausdorff measure *in its dimension* is undefined: i.e., the graph of f has Hausdorff dimension 2 with a zero 2-d Hausdorff measure. Hence, $\mathbb{E}[f]$ is undefined.

Moreover, observe that in §2.2, f is nowhere continuous and defined on a countably infinite set, which means depending on the enumeration of A or the sequence $\{a_r\}_{r=1}^{\infty}$, where the expected value of f (when it exists) is:

$$\mathbb{E}[f]=\lim_{t\to\infty}\frac{f(a_1)+f(a_2)+\cdots+f(a_t)}{t} \tag{3}$$

the expected value $\mathbb{E}[f]$ is any number from $\inf f$ to $\sup f$. Hence, we need a specific enumeration that gives a unique, satisfying, and finite expected value, generalizing this process to nowhere continuous functions defined on **uncountable** domains.

Thus, we want the "expected value of chosen sequences of bounded functions converging to f with the same satisfying and finite expected value" which we describe rigorously in later sections; however, consider the following definitions beginning with §2.3.1:

2.3.1. *Definition of sequences of Functions Converging to f.* Let $n\in\mathbb{N}$ and suppose function $f: A\subseteq\mathbb{R}^n\to\mathbb{R}$, where A and f are Borel.
The sequence of functions $(f_r)_{r\in\mathbb{N}}$, where $(A_r)_{r\in\mathbb{N}}$ is a sequence of sets and functions $f_r: A_r\to\mathbb{R}$, converges to f when:

For any $x\in A$, there exists a sequence $\mathbf{x}\in A_r$ *s.t.* $\mathbf{x}\to(x_1,\cdots,x_n)$ and $f_r(\mathbf{x})\to f(x_1,\cdots,x_n)$.

This is equivelant to:

$$(f_r,A_r)\to(f,A)$$

2.3.2. *Expected Value of Sequences of Functions Converging to f.* Hence, suppose:

- $(f_r,A_r)\to(f,A)$ (§2.3.1)
- $||\cdot||$ is the absolute value
- $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension
- $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra
- the integral is defined, w.r.t the Hausdorff measure in its dimension

The expected value of $(f_r)_{r\in\mathbb{N}}$ is a real number $\mathbb{E}[f_r]$, when the following is true:

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\left|\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(A_r)}\int_{A_r}f_r\,d\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}-\mathbb{E}[f_r]\right|\right|<\epsilon\right) \tag{4}$$

when no such $\mathbb{E}[f_r]$ exists, $\mathbb{E}[f_r]$ is infinite or undefined. (If the graph of f has zero Hausdorff measure in its dimension, replace $\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33].)

2.3.3. *The Set of All Bounded Functions/Sets.* Let $n\in\mathbb{N}$ and suppose the function $f: A\subseteq\mathbb{R}^n\to\mathbb{R}$, where A and f are Borel. Then, we define the following:

$\mathbf{B}(X)$ is the set of all bounded Borel subsets of the set X

$\mathfrak{B}(X)$ is the set of all bounded Borel functions with domain X

For example, $\mathbf{B}(\mathbb{R}^n)$ is the set of all bounded Borel subsets of $\mathbb{R}^n$ and $\mathfrak{B}(\mathbb{R})$ is the set of all bounded Borel functions on $\mathbb{R}$. Note, however:

Theorem 1. *For all $r,v\in\mathbb{N}$, suppose $A_r,B_v\in\mathbf{B}(\mathbb{R}^n)$, where $f_r\in\mathfrak{B}(A_r)$ and $g_v\in\mathfrak{B}(B_v)$. There exists a $f\in\mathbb{R}^A$, where $(f_r,A_r),(g_v,B_v)\to(f,A)$ and $\mathbb{E}[f_r]\neq\mathbb{E}[g_v]$* (§2.3.2)

For example, the expected values of the sequences of bounded functions converging to f (§2.3.1, §2.3.2) in §2.1 and §2.2 satisfy thm. 1. For simplicity, we illustrate this with §2.2.

2.3.4. *Example Illustrating Theorem 1.* For the second case of Borel $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ (§2.2), where $A = \mathbb{Q}$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \end{cases} \tag{5}$$

suppose:

$$(A_r)_{r\in\mathbb{N}} = (\{c/r! : c \in \mathbb{Z}, -r \cdot r! \leq c \leq r \cdot r!\})_{r\in\mathbb{N}}$$

and

$$(B_v)_{v\in\mathbb{N}} = (\{c/d : c \in \mathbb{Z}, d \in \mathbb{N}, d \leq v, -d \cdot v \leq c \leq d \cdot v\})_{v\in\mathbb{N}}$$

where for $f_r : A_r \to \mathbb{R}$,

$$f_r(x) = f(x) \text{ for all } x \in A_r \tag{6}$$

and for $g_v : B_v \to \mathbb{R}$

$$g_v(x) = f(x) \text{ for all } x \in B_v \tag{7}$$

Note, for all $r, v \in \mathbb{N}$:

- $\sup(A_r) = r$
- $\inf(A_r) = -r$
- $\sup(B_v) = v$
- $\inf(B_v) = -v$
- Since f is bounded, f_r and g_v are bounded

Hence, $A_r, B_v \in \mathbf{B}(\mathbb{R}^n)$, where $f_r \in \mathfrak{B}(A_r)$ and $g_v \in \mathfrak{B}(B_v)$. Also, the set-theoretic limit of $(A_r)_{r\in\mathbb{N}}$ and $(B_v)_{v\in\mathbb{N}}$ is $A = \mathbb{Q}$: i.e.,

$$\limsup_{r\to\infty} A_r = \bigcap_{r\geq 1} \bigcup_{q\geq r} A_q$$
$$\liminf_{r\to\infty} A_r = \bigcup_{r\geq 1} \bigcap_{q\geq r} A_q$$

where:

$$\limsup_{r\to\infty} A_r = \liminf_{r\to\infty} A_r = A = \mathbb{Q}$$
$$\limsup_{v\to\infty} B_v = \liminf_{v\to\infty} B_v = A = \mathbb{Q}$$

(We're unsure how to prove the set-theoretic limits; however, a mathematician specializing in limits should be able to check.)

Therefore, $(f_r, A_r), (g_v, B_v) \to (f, A)$ (thm. 1).

Now, suppose we want to average $(f_r)_{r\in\mathbb{N}}$ and $(g_v)_{v\in\mathbb{N}}$, which we denote $\mathbb{E}[f_r]$ and $\mathbb{E}[g_v]$. Note, this is the same as computing the following (i.e., the cardinality is $|\cdot|$ and the absolute value is $||\cdot||$):

$$\begin{aligned} &\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left|\left|\frac{1}{|A_r|}\int_{A_r} f \, d\mathcal{H}^0 - \mathbb{E}[f_r]\right|\right| < \epsilon\right) \implies \\ &\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left|\left|\frac{1}{|A_r|}\sum_{x\in A_r} f(x) - \mathbb{E}[f_r]\right|\right| < \epsilon\right) \end{aligned} \tag{8}$$

$$\begin{aligned} &\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(v \in \mathbb{N}) \left(v \geq N \Rightarrow \left|\left|\frac{1}{|B_v|}\int_{B_v} f \, d\mathcal{H}^0 - \mathbb{E}[g_v]\right|\right| < \epsilon\right) \implies \\ &\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(v \in \mathbb{N}) \left(v \geq N \Rightarrow \left|\left|\frac{1}{|B_v|}\sum_{x\in B_v} f(x) - \mathbb{E}[g_v]\right|\right| < \epsilon\right) \end{aligned} \tag{9}$$

Thus, if we assume $\mathbb{E}[f_r] = 1$ in eq. 8, using [8]:

The sum $\sum_{x\in A^\star} f(x)$ counts the number of fractions with an even denominator and an odd numerator in set $A^\star$, after canceling all possible factors of 2 in the fraction. Let us consider the first case. We can write:

$$\left|\left|1-|A_r|^{-1}\sum_{x\in A_r} f(x)\right|\right| = \left(|A_r| - \sum_{x\in A_r} f(x)\right)/|A_r| = H(r)/|A_r|$$

where $H(r)$ counts the fractions $x = c/r!$ in A_r that are not counted in $\sum_{x\in A^\star} f(x)$, i.e., for which $f(x) = 0$. This is the case when the denominator is odd after the cancellation of the factors of 2, i.e., when the numerator c has a number of factors of 2 greater than or equal to that of $r!$, which we will denote by $V(r) := v_2(r!)$ a.k.a the 2-valuation of $r!$, oeis:A11371$(r) = r - O(\ln(r))$ [10]. That means, c must be a multiple of $2^{V(r)}$. The number of such c with $-r\cdot r! \le c \le r \cdot r$ is simply the length of that inteval, equal to $|A_r| = 2r(r!)+1$, divided by $2^{V(r)}$. Thus,

$$\left|\left|1-|A_r|^{-1}\sum_{x\in A_r} f(x)\right|\right| = [|A_r|/2^{V(r)}]/|A_r| \sim 1/2^{V(r)} = 1/2^{n-O(\log n)}$$

This obviously tends to zero, proving $\mathbb{E}[f_r] = 1$

Last, we need to show $\mathbb{E}[g_v] = 1/3$ in eq. 9, where $\mathbb{E}[f_r] \neq \mathbb{E}[g_v]$, proving theorem 1.

Concerning the second case [8], it is again simpler to consider the complementary set of $x \in B_v$ such that the denominator is odd when all possible factors of 2 are canceled. We can see that for $v = 2p-1$, and these obviously include all those we had for smaller v. The "new" elements in B_v with $v = 2p-1$ are those that have the denominator $d = 2p-1$ when written in lowest terms. Their number is equal to the number of $\kappa < d$, $\gcd(\kappa, d) = 1$, which is given by Euler's ϕ function. Since we also consider negative fractions, we have to multiply this by 2. Including $x = 0$, we have $G(v) = |\{x \in B_v | f(x) = 0\}| = 1 + 2\sum_{0\le\kappa\le v/2}\phi(2\kappa+1)$. There is no simple explicit expression for this (cf. oeis:A99957 [11]), but we know that $G(v) = 1 + 2\cdot\text{A99957}(v/2) \sim 2\cdot 8(v/2)^2/\pi^2 = 4v^2/\pi^2$ [11]. On the other hand, the total number of all elements of B_v is $|B_v| = 1 + 2\sum_{1\le\kappa\le v}\phi(\kappa)$, since each time we increase v by 1, we have the additional fractions with the new denominator $d = v$ and the numerators are coprime with d, again with the sign + or −. From oeis:A002088 [9] we know that $\sum_{1\le\kappa\le v}\phi(\kappa) = 3v^2/\pi^2 + O(v\log v)$, so $|B_v| \sim 6v^2/\pi^2$, which finally gives $|B_v|^{-1}\sum_{x\in B_v} f(x) = (|B_v| - G(v))/|B_v| \sim (6-4)/6 = 1/3$ as desired.

Hence, $\mathbb{E}[g_v] = 1/3$ and $\mathbb{E}[f_r] \neq \mathbb{E}[g_v]$ proving thm. 1. Thus, consider:

2.4. **Definition of Prevalent and Shy Sets.** A Borel set $E \subset X$ is said to be **prevalent** if there exists a Borel measure μ on X such that:

1. $0 < \mu(C) < \infty$ for some compact subset C of X, and
2. the set $E + x$ has full μ-measure (that is, the complement of $E + x$ has measure zero) for all $x \in X$.

More generally, a subset F of X is prevalent if F contains a prevalent Borel Set.

Moreover:

- The complement of a prevelant set is a **shy** set.

Hence:

- If $F \subset X$ is *prevelant*, we say "**almost every**" element of X lies in F.
- If $F \subset X$ is *shy*, we say "**almost no**" element of X lies in F.

2.5. **Motivation for Averaging** §2.1 **and** §2.2. If $\mathbb{E}[f]$ is the expected value of f, w.r.t the Hausdorff measure *in its dimension*,

$$\mathbb{E}[f] = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A)}(A)}\int_A f\, d\mathcal{H}^{\dim_{\mathrm{H}}(A)} \tag{10}$$

Consider the following problems:

Theorem 2. *If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where $\mathbb{E}[f]$ is finite, then F is shy* (§2.4).

Note 3 (**Proof theorem 2 is true**). *We follow the argument presented in Example 3.6 of this paper* [12], *take* $X := L^0(A)$ *(measurable functions over* A*), let* P *denote the one-dimensional subspace of* A *consisting of constant functions (assuming the Lebesgue measure on* A*) and let* $F := L^0(A) \setminus L^1(A)$ *(measurable functions over* A *without finite integral). Let* λ_P *denote the Lebesgue measure over* P*, for any fixed* $f \in F$*:*

$$\lambda_P\left(\left\{\alpha \in \mathbb{R}\middle| \int_A (f+\alpha)d\mu < \infty\right\}\right) = 0$$

Meaning P *is a one-dimensional, so* f *is a 1-prevalent set.*

Note 4 (**Way of Approaching Theorem 2**). *For all* $r \in \mathbb{N}$*, suppose that* $A_r \in \mathbf{B}(\mathbb{R}^n)$ *and* $f_r \in \mathfrak{B}(A_r)$*. If* $F \subset \mathbb{R}^A$ *is the set of all* $f \in \mathbb{R}^A$*, where there exists* $A_r \in \mathbf{B}(\mathbb{R}^n)$ *and* $f_r \in \mathfrak{B}(A_r)$ *such that* $(f_r, A_r) \to (f, A)$ *and* $\mathbb{E}[f_r]$ *is finite* (§2.3.2)*, then* F *should be prevalent* (§2.4) *or neither prevalent nor shy* (§2.4).

Theorem 5. *For all* $r, v \in \mathbb{N}$*, suppose* $A_r, B_v \in \mathbf{B}(\mathbb{R}^n)$*, where* $f_r \in \mathfrak{B}(A_r)$ *and* $g_v \in \mathfrak{B}(B_v)$*. When* $F \subset \mathbb{R}^A$ *is the set of all* $f \in \mathbb{R}^A$*, where* $(f_r, A_r), (g_v, B_v) \to (f, A)$ *and* $\mathbb{E}[f_r] \neq \mathbb{E}[g_v]$*, then* F *is prevalent* (§2.4).

Note 6 (**Possible method to proving theorem 5 true**). *For all* $r, v \in \mathbb{N}$*, suppose* $A_r, B_v \in \mathbf{B}(\mathbb{R}^n)$ *where* $f_r \in \mathfrak{B}(A_r)$ *and* $g_v \in \mathfrak{B}(B_v)$*. Therefore, suppose* $Q \subset \mathbb{R}^A$ *is the set of all* $f \in \mathbb{R}^A$ *whose lines of symmetry intersect at one point, where if* $(f_r, A_r), (g_v, B_v) \to (f, A)$*, then* $\mathbb{E}[f_r] = \mathbb{E}[g_v]$*. In addition,* $Q' \subset \mathbb{R}^A$ *is the set of symmetric* $f \in \mathbb{R}^A$ *which clearly forms a shy subset of* $\mathbb{R}^A$*. Since* $Q \subset Q'$*, we have proven that* Q *is also shy (i.e., a subset of a shy set is also shy). Since the complement of the shy set* Q *is prevalent,* $F = \mathbb{R}^A \setminus Q$ *is prevalent, such that for all* $f \in F$*,* $(f_r, A_r), (g_v, A_v) \to (f, A)$ *and* $\mathbb{E}[f_r] \neq \mathbb{E}[g_v]$*. If this is correct, we have partially proven thm.* 5.

Note 7 (**Way of Approaching Theorem 5**). *For all* $r \in \mathbb{N}$*, suppose* $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$ *is an arbitrary set, where* $A_r \in \mathcal{B}$ *and* $\mathscr{B} \subset \mathfrak{B}(A_r)$ *is an arbitrary set such that* $f_r \in \mathscr{B}$*. If* $F \subset \mathbb{R}^A$ *is the set of all* $f \in \mathbb{R}^A$*, where* $(f_r, A_r) \to (f, A)$ *and* $\mathbb{E}[f_r]$ *is unique, then* F *should be prevelant* (§2.4).

Since thm. 2 and 5 are true, we need to solve both theorems *at once* with the following:

2.5.1. Approach.

> For all $r \in \mathbb{N}$, suppose $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$ is an arbitrary set, where $A_r \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r)$ is an arbitrary set such that $f_r \in \mathscr{B}$. If $F \subset \mathbb{R}^A$ is the collection of all $f \in \mathbb{R}^A$, where $(f_r, A_r) \to (f, A)$ and $\mathbb{E}[f_r]$ is unique, satisfying (§3) and finite, then F should be:
> 1. a prevalent (§2.4) subset of $\mathbb{R}^A$
> 2. If not prevalent (§2.4) then neither prevalent (§2.4) nor shy (§2.4) subset of $\mathbb{R}^A$.

3. ATTEMPT TO DEFINE “SATISFYING” IN THE APPROACH OF §2.5.1

3.1. **Leading Question.** To define *satisfying* in the blockquote of the §2.5.1, we ask the **leading question**...

> Suppose, for all $r, v \in \mathbb{N}$, there exists arbitrary set $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, where $A_r^\star \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r^\star)$ (§2.3.3) such that:
> (A) $f_r^\star \in \mathscr{B}$
> (B) $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$ and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star}) \cup (\mathfrak{B}(A_r^\star) \setminus \mathscr{B})$
> (C) $(G_r^\star)_{r\in\mathbb{N}} = (\text{graph}(f_r^\star))_{r\in\mathbb{N}}$ is the sequence of the graph of each $f_r^\star$ (§2.3.1)
> (D) $\Box$ is the logical symbol for “it's necessary”
> (E) C is a reference point in $\mathbb{R}^{n+1}$ (e.g., the origin)
> (F) E is the fixed, expected rate of expansion of $(G_r^\star)_{r\in\mathbb{N}}$ w.r.t a reference point C: e.g., $E = 1$ (§3.1.C, §3.1.E)
> (G) $\mathcal{E}(C, G_r^\star)$ is the actual rate of expansion of $(G_r^\star)_{r\in\mathbb{N}}$ w.r.t a reference point C (§3.1.C, §3.1.E, §5.4)
>
> Does there exist an unique choice function, which for all $r \in \mathbb{N}$, chooses an unique set $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, where $A_r^\star \in \mathcal{B}$ and a unique set $\mathscr{B} \subset \mathfrak{B}(A_r^\star)$ such that $f_r^\star \in \mathscr{B}$, where:
> (1) $(f_r^\star, A_r^\star) \to (f, A)$ (§2.3.1)

(2) For all $v \in \mathbb{N}$, where for all $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$ and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star}) \cup (\mathfrak{B}(A_r^\star) \setminus \mathscr{B})$, assuming $(f_v^{\star\star}, A_v^{\star\star}) \to (f, A)$, the "measure" (§5.3.1, §5.3.2) of $(G_r^\star)_{r\in\mathbb{N}} = (\text{graph}(f_r^\star))_{r\in\mathbb{N}}$ (§3.1.C) must increase at a rate linear or superlinear to that of $(G_v^{\star\star})_{v\in\mathbb{N}} = (\text{graph}(f_v^{\star\star}))_{v\in\mathbb{N}}$ (§3.1.C)

(3) $\mathbb{E}[f_r^\star]$ is unique and finite (§2.3.2)

(4) For some $A_r^\star \in \mathcal{B}$ and $f_r^\star \in \mathscr{B}$ satisfying (1), (2) and (3), when f is unbounded (i.e, skip (4) when f is bounded), for all $s \in \mathbb{N}$ and for any set $\mathcal{B}' \subset \mathbf{B}(\mathbb{R}^n)$, where $A_s^{\star\star\star} \in \mathcal{B}'$, and for any set $\mathscr{B}' \subset \mathfrak{B}(A_s^{\star\star\star})$, where $\star \mapsto \star\star\star$, $r \mapsto s$, $\mathcal{B} \mapsto \mathcal{B}'$, and $\mathscr{B} \mapsto \mathscr{B}'$ in (1), (2) and (3), s.t. $\neg\Box(\mathbb{E}[f_r^\star] = \mathbb{E}[f_s^{\star\star\star}])$ (§2.3.1, §2.3.2, §3.1.D), when $f_s^{\star\star\star} \in \mathscr{B}'$ satisfies (1), (2) and (3):
   - If the absolute value is $||\cdot||$ and the $(n+1)$-th coordinate of C (§3.1.E) is x_{n+1}, $||\mathbb{E}[f_r^\star] - x_{n+1}|| \leq ||\mathbb{E}[f_s^{\star\star\star}] - x_{n+1}||$ (§2.3.1, §2.3.2)
   - If $r \in \mathbb{N}$, then for all linear $s_1 : \mathbb{N} \to \mathbb{N}$, where $s = s_1(r)$ and the Big-O notation is $\mathcal{O}$, there exists a function $K : \mathbb{R} \to \mathbb{R}$, where the absolute value is $||\cdot||$ and (§3.1.F-G):

     $$\begin{aligned} ||\mathcal{E}(C, G_r^\star) - E|| =& \mathcal{O}(K(||\mathcal{E}(C, G_s^{\star\star\star}) - E||)) \\ =& \mathcal{O}(K(||\mathcal{E}(C, G_{s_1(r)}^{\star\star\star}) - E||)) \end{aligned}$$

     such that:

     $$0 \leq \lim_{x\to+\infty} K(x)/x < +\infty$$

     In simpler terms, "the rate of divergence" of $||\mathcal{E}(C, G_r^\star) - E||$ (§3.1.F-G) is *less than or equal* to "the rate of divergence" of $||\mathcal{E}(C, G_s^{\star\star\star}) - E||$ (§3.1.F-G).

(5) When set $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where a choice function chooses a collection $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, where $A_r^\star \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r^\star)$ such that $f_r^\star \in \mathscr{B}$ satisfies (1), (2), (3) and (4), then F should be:
   (a) a prevelant (§2.4) subset of $\mathbb{R}^A$
   (b) If not (a), then neither a prevalent (§2.4) nor shy (§2.4) subset of $\mathbb{R}^A$

(6) Out of all choice functions which satisfy (1), (2), (3), (4) and (5), we choose the one with the simplest form, meaning for each choice function fully expanded, we take the one with the fewest variables/numbers?

(In case this is unclear, see §5.) We are convinced $\mathbb{E}[f_r^\star]$ in (§3.1 crit. 3) isn't *unique* nor *satisfying enough* to answer the approach of §2.5.1. Still, adjustments are possible by changing the criteria or by adding new criteria to *the question*.

4. QUESTION REGARDING MY WORK

Most don't have time to address everything in my research, hence I ask the following:

> *Is there a research paper which already solves the ideas I'm woring on? (Non-published papers, such as mine* [6]*, don't count.)*

5. CLARIFYING §3

See §3.1 once reading §5, and consider the following:

> Is there a simpler version of the definitions below?

5.1. Example of sequences of Bounded Functions Converging to f (§2.3.1).

The sequence of bounded functions $(f_r)_{r\in\mathbb{N}}$, where $(A_r)_{r\in\mathbb{N}}$ is a sequence of bounded sets and function $f_r : A_r \to \mathbb{R}$, converges to Borel $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ when:

For any $x \in A$ there exists a sequence $\mathbf{x} \in A_r$ s.t. $\mathbf{x} \to (x_1, \cdots, x_n)$ and $f_r(\mathbf{x}) \to f(x_1, \cdots, x_n)$

This is equivalent to:

$$(f_r, A_r) \to (f, A)$$

Example 0.1 (**Example of §2.3.1**). *If $A = \mathbb{R}$ and $f : A \to \mathbb{R}$, where $f(x) = 1/x$, then an example of $(f_r)_{r\in\mathbb{N}}$, such that $f_r : A_r \to \mathbb{R}$ is:*

(1) $(A_r)_{r\in\mathbb{N}} = ([-r, -1/r] \cup [1/r, r])_{r\in\mathbb{N}}$
(2) $f_r(x) = 1/x$ *for* $x \in A_r$

Example 0.2 (**More Complex Example**)**.** *If $A = \mathbb{R}$ and $f : A \to \mathbb{R}$, where $f(x) = x$, then an example of $(f_r)_{r\in\mathbb{N}}$, such that $f_r : A_r \to \mathbb{R}$ is:*

1. $(A_r)_{r\in\mathbb{N}} = ([-r, r])_{r\in\mathbb{N}}$
2. $f_r(x) = x + (1/r)\sin(x)$ *for* $x \in A_r$

5.2. **Expected Value of Bounded Sequence of Functions.** The expected value of $(f_r)_{r\in\mathbb{N}}$ is a real number $\mathbb{E}[f_r]$, when the following is true:

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(A_r)}\int_{A_r} f_r \, d\mathcal{H}^{\dim_{\mathrm{H}}(A_r)} - \mathbb{E}[f_r]\right|\right| < \epsilon\right) \tag{11}$$

otherwise when no such $\mathbb{E}[f_r]$ exists, $\mathbb{E}[f_r]$ is infinite or undefined. (If the graph of f has zero Hausdorff measure in its dimension, replace $\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33].)

5.2.1. *Example.* Using example 0.1, when $(f_r)_{r\in\mathbb{N}} = (\{(x, 1/x) : x \in [-r, -1/r] \cup [1/r, r]\})_{r\in\mathbb{N}}$ where:

1. $(A_r)_{r\in\mathbb{N}} = ([-r, -1/r] \cup [1/r, r])_{r\in\mathbb{N}}$
2. $f_r(x) = 1/x$ for $x \in A_r$

If we assume $\mathbb{E}[f_r] = 0$:

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(A_r)}\int_{A_r} f_r \, d\mathcal{H}^{\dim_{\mathrm{H}}(A_r)} - \mathbb{E}[f_r]\right|\right| < \epsilon\right) \tag{12}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\Big(r \geq N \Rightarrow \tag{13}$$

$$\left|\left|\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}([-r,-1/r]\cup[1/r,r])}([-r, -1/r] \cup [1/r, r])}\int_{[-r,-1/r]\cup[1/r,r]} 1/x \, d\mathcal{H}^{\dim_{\mathrm{H}}([-r,-1/r]\cup[1/r,r])} - 0\right|\right| < \epsilon\Big) \tag{14}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{\mathcal{H}^{1}([-r, -1/r] \cup [1/r, r])}\int_{[-r,-1/r]\cup[1/r,r]} 1/x \, d\mathcal{H}^{1}\right|\right| < \epsilon\right) \tag{15}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{(-1/r - (-r)) + (r - 1/r)}\left(\int_{-r}^{-1/r} 1/x \, dx + \int_{1/r}^{r} 1/x \, dx\right)\right|\right| < \epsilon\right) \tag{16}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{(r - 1/r) + (-1/r + r)}\left(\ln(||x||) + C\Big|_{-r}^{-1/r} + \ln(||x||) + C\Big|_{1/r}^{r}\right)\right|\right| < \epsilon\right) \tag{17}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{(r - 1/r) + (-1/r + r)}\left(\ln(|| - r||) - \ln(|| - 1/r||) + \ln(||r||) - \ln(||1/r||)\right)\right|\right| < \epsilon\right) \tag{18}$$

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\left|\frac{1}{2r - 2/r} \cdot 4\ln(r)\right|\right| < \epsilon\right) \tag{19}$$

To prove eq. 19 is true, recall:

$$r \ll e^{r/2}, \, e^{1/r} \ll e^{r} \tag{20}$$

$$r \ll e^{r/2}, \, e^{1/(2r)} \ll e^{r/2} \tag{21}$$

$$re^{1/(2r)} \ll e^{r/2} \tag{22}$$

$$r \ll e^{r/2}/e^{1/(2r)} \tag{23}$$

$$r \ll e^{r/2-1/(2r)} \tag{24}$$

$$\ln(r) \ll r/2 - 1/(2r) \tag{25}$$

$$4\ln(r) \ll 2r - 2/r \tag{26}$$

Hence, for all $\varepsilon > 0$

$$4\ln(r) < \varepsilon(2r - 2/r) \tag{27}$$

$$\frac{4\ln(r)}{2r - 2/r} < \varepsilon \tag{28}$$

$$\left|\left|\frac{4\ln(r)}{2r - 2/r}\right|\right| < \varepsilon \tag{29}$$

Since eq. 19 is true, $\mathbb{E}[f_r]=0$. Note, if we simply took the average of f from $(-\infty,\infty)$, using the improper integral, the expected value:

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4-x_3)+(x_2-x_1)}\left(\int_{x_1}^{x_2}\frac{1}{x}\,dx+\int_{x_3}^{x_4}\frac{1}{x}\,dx\right)= \tag{30}$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4-x_3)+(x_2-x_1)}\left(\ln(||x||)+C\Big|_{x_1}^{x_2}+\ln(||x||)+C\Big|_{x_3}^{x_4}\right)= \tag{31}$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4-x_3)+(x_2-x_1)}\left(\ln(||x_2||)-\ln(||x_1||)+\ln(||x_4||)-\ln(||x_3||)\right) \tag{32}$$

is $+\infty$ (when $x_2=1/x_1$, $x_3=1/x_4$, and $x_1=\exp\left(x_4^2\right)$) or $-\infty$ (when $x_2=1/x_1$, $x_3=1/x_4$, and $x_4=-\exp\left(x_1^2\right)$), making $\mathbb{E}[f]$ undefined. (However, using eq. 12-19, we get the $\mathbb{E}[f_r]=0$ instead of an undefined value.)

5.3. Defining the “Measure”.

5.3.1. *Preliminaries.* We define the “**measure**” of $(G_r^\star)_{r\in\mathbb{N}}$, in §5.3.2, which is the sequence of the graph of each $f_r^\star$ (§3.1.C). To understand this “measure”, continue reading.

(1) For every $r\in\mathbb{N}$, “over-cover” $G_r^\star$ with minimal, pairwise disjoint sets of equal $\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}$ measure. (We denote the equal measures ε, where the former sentence is defined $\mathbf{C}(\varepsilon,G_r^\star,\omega)$: i.e., $\omega\in\Omega_{\varepsilon,r}$ enumerates all collections of these sets covering $G_r^\star$. In case this step is unclear, see §8.1. Moreover, when there exists a $r\in\mathbb{N}$, where $\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}(G_r^\star)=0$, replace the Hausdorff measure $\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi_{h,g}^{\mu}(q,t)}$ [1, p.26-33].)

(2) For every ε, r and ω, take a sample point from each set in $\mathbf{C}(\varepsilon,G_r^\star,\omega)$. The set of these points is “the sample” which we define $\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)$: i.e., $\psi\in\Psi_{\varepsilon,r,\omega}$ enumerates all possible samples of $\mathbf{C}(\varepsilon,G_r^\star,\omega)$. (If this is unclear, see §8.2.)

(3) For every ε, r, ω and ψ,

  (a) Take a “pathway” of line segments: we start with a line segment from arbitrary point x_0 of $\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)$ to the sample point with the smallest $(n+1)$-dimensional Euclidean distance to x_0 (i.e., when more than one sample point has the smallest $(n+1)$-dimensional Euclidean distance to x_0, take either of those points). Next, repeat this process until the “pathway” intersects with every sample point once. (In case this is unclear, see §8.3.1.)

  (b) Take the set of the length of all segments in (a), except for lengths that are outliers (i.e., for any constant $C>0$, the outliers are more than C times the interquartile range of the length of all line segments as $r\to\infty$ or $\varepsilon\to 0$). Define this $\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi))$. (If this is unclear, see §8.3.2.)

  (c) Multiply remaining lengths in the pathway by a constant so they add up to one (i.e., a probability distribution). This will be denoted $\mathbb{P}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)))$. (In case this is unclear, see §8.3.3)

  (d) Take the shannon entropy [7, p.61-95] of step (c). We define this:

$$\mathrm{E}(\mathbb{P}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi))))=\sum_{x\in\mathbb{P}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)))}-x\log_2 x$$

  which will be *shortened* to $\mathrm{E}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)))$. (If this is unclear, see §8.3.4.)

  (e) Maximize the entropy w.r.t all ”pathways”. This we will denote:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)))=\sup_{x_0\in\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)}\mathrm{E}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)))$$

  (In case this is unclear, see §8.3.5.)

(4) Therefore, the **maximum entropy**, using (1) and (2) is:

$$\mathrm{E}_{\max}(\varepsilon,r)=\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^\star,\omega),\psi)))$$

5.3.2. *What Am I Measuring?* We define $(G_r^\star)_{r\in\mathbb{N}}$ and $(G_v^{\star\star})_{v\in\mathbb{N}}$, which respectively are sequences of the graph for each of the bounded functions $f_r^\star$ and $f_v^{\star\star}$ (§3.1.C). Hence, for **constant** ε and *cardinality* $|\cdot|$

(a) Using (2) and (3e) of section 5.3.1, suppose:

$\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi)|} =$

$\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star\star}_v, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star\star}_v, \omega'), \psi'))) \geq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi)))\right\}$

then (using $\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi)|}$) we get:

$$\overline{\alpha}\,(\varepsilon, r, \omega, \psi) = \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi)|} / \, |\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi))|$$

(b) Also, using (2) and (3e) of section 5.3.1, suppose:

$\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi)|} =$

$\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star\star}_v, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star\star}_v, \omega'), \psi'))) \leq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi)))\right\}$

then (using $\underline{|\mathcal{S}(\mathbf{C}(\epsilon, G^{\star}_r, \omega), \psi)|}$) we also get

$$\underline{\alpha}\,(\varepsilon, r, \omega, \psi) = \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi)|} / \, |\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi))|$$

(1) If using $\overline{\alpha}\,(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}\,(\epsilon, r, \omega, \psi)$ we have:

$$1 < \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}\,(\varepsilon, r, \omega, \psi)\,, \liminf_{\varepsilon \to 0} \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}\,(\varepsilon, r, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(G^{\star}_r)_{r\in\mathbb{N}}$ increases at a rate **superlinear** to that of $(G^{\star\star}_v)_{v\in\mathbb{N}}$.

(2) If using equations $\overline{\alpha}\,(\varepsilon, v, \omega, \psi)$ and $\underline{\alpha}\,(\varepsilon, v, \omega, \psi)$ (where, using $\overline{\alpha}\,(\varepsilon, r, \omega, \psi)$ and $\underline{\alpha}\,(\varepsilon, r, \omega, \psi)$, we swap r with v and $G^{\star}_r$ with $G^{\star\star}_v$) we get:

$$1 < \limsup_{\varepsilon \to 0} \limsup_{v \to \infty} \sup_{\omega \in \Omega_{\varepsilon,v}} \sup_{\psi \in \Psi_{\varepsilon,v,\omega}} \overline{\alpha}\,(\varepsilon, v, \omega, \psi)\,, \liminf_{\varepsilon \to 0} \liminf_{v \to \infty} \inf_{\omega \in \Omega_{\varepsilon,v}} \inf_{\psi \in \Psi_{\varepsilon,v,\omega}} \underline{\alpha}\,(\varepsilon, v, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(G^{\star}_r)_{r\in\mathbb{N}}$ increases at a rate **sublinear** to that of $(G^{\star\star}_v)_{v\in\mathbb{N}}$.

(3) If using equations $\overline{\alpha}\,(\varepsilon, r, \omega, \psi)$, $\underline{\alpha}\,(\varepsilon, r, \omega, \psi)$, $\overline{\alpha}\,(\varepsilon, v, \omega, \psi)$, and $\underline{\alpha}\,(\varepsilon, v, \omega, \psi)$, we **both** have:

  (a) $\limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}\,(\varepsilon, r, \omega, \psi)$ or $\liminf_{\varepsilon \to 0} \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}\,(\varepsilon, r, \omega, \psi)$ are equal to zero, one or $+\infty$

  (b) $\limsup_{\varepsilon \to 0} \limsup_{v \to \infty} \sup_{\omega \in \Omega_{\varepsilon,v}} \sup_{\psi \in \Psi_{\varepsilon,v,\omega}} \overline{\alpha}\,(\varepsilon, v, \omega, \psi)$ or $\liminf_{\varepsilon \to 0} \liminf_{v \to \infty} \inf_{\omega \in \Omega_{\varepsilon,v}} \inf_{\psi \in \Psi_{\varepsilon,v,\omega}} \underline{\alpha}\,(\varepsilon, v, \omega, \psi)$ are equal to zero, one or $+\infty$

then *what I'm measuring from* $(G^{\star}_r)_{r\in\mathbb{N}}$ increases at a rate **linear** to that of $(G^{\star\star}_v)_{v\in\mathbb{N}}$.

5.3.3. *Example of The "Measure" of* $(G^{\star}_r)$ *Increasing at Rate Super-linear to that of* $(G^{\star\star}_v)$. Suppose, we have function $f : A \to \mathbb{R}$, where $A = \mathbb{Q} \cap [0, 1]$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \end{cases} \tag{33}$$

such that:

$$(A^{\star}_r)_{r\in\mathbb{N}} = (\{c/r! : c \in \mathbb{Z}, 0 \leq c \leq r!\})_{r\in\mathbb{N}}$$

and

$$(A^{\star\star}_v)_{v\in\mathbb{N}} = (\{c/d : c \in \mathbb{Z}, d \in \mathbb{N}, d \leq v, 0 \leq c \leq v\})_{v\in\mathbb{N}}$$

where for $f^{\star}_r : A^{\star}_r \to \mathbb{R}$,

$$f^{\star}_r(x) = f(x) \text{ for all } x \in A^{\star}_r \tag{34}$$

and $f^{\star\star}_v : A^{\star\star}_v \to \mathbb{R}$

$$f^{\star\star}_v(x) = f(x) \text{ for all } x \in A^{\star\star}_v \tag{35}$$

Hence, when $(G^{\star}_r)_{r\in\mathbb{N}}$ is:

$$(G^{\star}_r)_{r\in\mathbb{N}} = (\{(x, f^{\star}_r(x)) : x \in A^{\star}_r\})_{r\in\mathbb{N}} \tag{36}$$

and $(G^{\star\star}_v)_{v\in\mathbb{N}}$ is:

$$(G^{\star\star}_v)_{v\in\mathbb{N}} = (\{(x, f^{\star\star}_v(x)) : x \in A^{\star\star}_v\})_{v\in\mathbb{N}} \tag{37}$$

Note, the following:

Since $\varepsilon > 0$ and $A = \mathbb{Q} \cap [0, 1]$ is countably infinite, there exists minimum ε which is 1. Therefore, we don't need $\varepsilon \to 0$. We also maximize $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G^{\star}_r, \omega), \psi)))$ (§5.3.1 step 3e) by the following procedure:

(1) For every $r \in \mathbb{N}$, group $(x, y) \in G_r^\star$ into $(x, f_r^\star(x))$, where x has an even denominator when simplified: i.e.,

$$S_{1,r} = \{(x, f_r^\star(x)) : x \in A_r^\star \cap \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1]\}$$

then group $(x, y) \in G_r^\star$ into $(x, f_r^\star(x))$, where x has an odd denominator when simplified: i.e.,

$$S_{2,r} = \{(x, f_r^\star(x)) : x \in A_r^\star \cap (\mathbb{Q} \setminus \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\}) \cap [0,1]\}$$

(2) Arrange the points in $S_{1,r}$ from least to greatest and take the 2-d Euclidean distance between each pair of consecutive points in $S_{1,r}$. In this case, since the points lie on $y = 1$, take the absolute difference between the x-coordinates of $S_{1,r}$, and then call this $\mathcal{D}_{1,r}$. (Note, this is similar to §5.3.1 step 3a).

(3) Repeat step (2) for $S_{2,r}$, then call this $\mathcal{D}_{2,r}$. (Note, all point of $S_{2,r}$ lie on $y = 0$.)

(4) Remove any outliers from $\mathcal{D}_r = \mathcal{D}_{1,r} \cup \mathcal{D}_{2,r} \cup \{d((\frac{r!-1}{r!}, 1), (1, 0))\}$ (i.e., d is the 2-d Euclidean distance between points $(\frac{r!-1}{r!}, 1)$ and $(1, 0)$). Note, in this case, $\mathcal{D}_{2,r}$ and $\{d((\frac{r!-1}{r!}, 1), (1, 0))\}$ should be outliers (i.e., for any $C > 0$, the lengths of $\mathcal{D}_{2,r}$ and $\{d((\frac{r!-1}{r!}, 1), (1, 0))\}$ are more than C times the interquartile range of the lengths of $\mathcal{D}_r$ as $r \to \infty$) leaving us with $\mathcal{D}_{1,r}$.

(5) Multiply the remaining lengths in the pathway by a constant so they add up to one. (See `P[r]` of code 1 for an example)

(6) Take the entropy of the probability distribution. (See `entropy[r]` of code 1 for an example.)

We can illustrate this process with the following code:

Code 1. Illustration of step (1)-(6)

```
(*We're using Mathematica*)

Clear["*Global‘*"]
A[r_] := A[r] = Range[0, r!]/(r!)

(*Below is step 1*)
S1[r_] :=
 S1[r] = Sort[Select[A[r], Boole[IntegerQ[Denominator[#]/2]] == 1 &]]
S2[r_] :=
 S2[r] = Sort[Select[A[r], Boole[IntegerQ[Denominator[#]/2]] == 0 &]]

(*Below is step 2*)
Dist1[r_] := Dist1[r] = Differences[S1[r]]

(*Below is step 3*)
Dist2[r_] := Dist2[r] = Differences[S2[r]]

(*Below is step 4*)
NonOutliers[r_] :=
 NonOutliers[r] = Dist1[r] (*We exclude Dist2[r] since it's an outlier*)

(*Below is step 5*)
P[r_] := P[r] = NonOutliers[r]/Total[NonOutliers[r]]

(*Below is step 6*)
entropy[r_] := entropy[r] = Total[-P[r] Log[2, P[r]]]
```

Taking `Table[{r,entropy[r]},{r,3,8}]`, we get:

Code 2. Output of `Table[{r,entropy[r]},{r,3,8}]`

```
Clear["*Global‘*"]
{{{3,1}, {4,(2 Log[11])/(11 Log[2]) + (9 Log[22])/(11 Log[2])},
  {5,(14 Log[59])/(59 Log[2]) + (45 Log[118])/(59 Log[2])},
  {6,(44 Log[359])/(359 Log[2]) + (315 Log[718])/(359 Log[2])},
  {7,(314 Log[2519])/(2519 Log[2]) + (2205 Log[5038])/(2519 Log[2])},
  {8,(314 Log[20159])/(20159 Log[2]) + (19845 Log[40318])/(20159 Log[2])}}}
```

and notice when:

(1) $c(r) = (r!)/2 - 1$

(2) $\{b(4) \mapsto 9,\ b(5) \mapsto 45,\ b(6) \mapsto 315,\ b(7) \mapsto 2205,\ b(8) \mapsto 19845\}$

(3) $a(r) + b(r) = c(r)$

the output of code 2 can be defined:

$$\frac{a(r)\log_2(c(r))}{c(r)}+\frac{b(r)\log(2c(r)))}{c(r)}=\frac{a(r)\log_2(c(r))+b(r)\log(2c(r))}{c(r)} \tag{38}$$

Hence, since $a(r) = c(r) - b(r) = (r!)/2 - 1 - b(r)$:

$$\frac{a(r)\log_2(c(r))+b(r)\log(2c(r))}{c(r)}= \tag{39}$$

$$\frac{(r!/2-1-b(r))\log_2(c(r))+b(r)\log_2(2c(r))}{c(r)}= \tag{40}$$

$$\frac{(r!/2)\log_2(c(r))-\log_2(c(r))-b(r)\log_2(r)+b(r)\log_2(c(r))+b(r)\log_2(2)}{c(r)}= \tag{41}$$

$$\frac{(r!/2)\log_2(c(r))-\log_2(c(r))+b(r)}{c(r)}= \tag{42}$$

$$\frac{(r!/2-1)\log_2(c(r))+b(r)}{c(r)}= \tag{43}$$

$$\frac{(r!/2-1)\log_2(r!/2-1)+b(r)}{r!/2-1}= \tag{44}$$

$$\log_2(r!/2-1)+\frac{b(r)}{r!/2-1}= \tag{45}$$

and $\lim_{r\to\infty} b(r)/c(r) = 1$ (I need help proving this):

$$\log_2(r!/2-1)+\frac{b(r)}{r!/2-1}\sim\log_2(r!/2-1)+1 \tag{46}$$

$$\log_2(r!/2-1)+\log_2(2)= \tag{47}$$

$$\log_2(2(r!/2-1)) \tag{48}$$

$$\log_2(r!-2)\sim\log_2(r!) \tag{49}$$

Hence, `entropy[r]` is the same as:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1,G_r^{\star},\omega),\psi)))\sim\log_2(r!) \tag{50}$$

Now, repeat code 1 with:

$$(G_v^{\star\star})_{v\in\mathbb{N}}=\{(x,f_v^{\star\star}(x)):x\in A_v^{\star\star}:=\{c/d:c\in\mathbb{Z},d\in\mathbb{N},d\le v,0\le c\le v\}\}_{v\in\mathbb{N}}$$

CODE 3. Illustration of step (1)-(6) on $(G_v^{\star\star})$

```
(*We're using Mathematica*)

Clear["*Global`*"]
A[v_] := A[v] =
   DeleteDuplicates[Flatten[Table[Range[0, t]/t, {t, 1, v}]]]

(*Below is step 1*)
S1[v_] :=
 S1[v] = Sort[Select[A[v], Boole[IntegerQ[Denominator[#]/2]] == 1 &]]
S2[v_] :=
 S2[v] = Sort[Select[A[v], Boole[IntegerQ[Denominator[#]/2]] == 0 &]]

(*Below is step 2*)
Dist1[v_] := Dist1[v] = Differences[S1[v]]

(*Below is step 3*)
Dist2[v_] := Dist2[v] = Differences[S2[v]]

(*Below is step 4*)
NonOutliers[v_] :=
 NonOutliers[v] = Join[Dist1[v], Dist2[v]] (*There are no outliers*)
```

```
(*Below is step 5*)
P[v_] := P[v] = NonOutliers[v]/Total[NonOutliers[v]]

(*Below is step 6*)

entropy[v_] := entropy[v] = N[Total[-P[v] Log[2, P[v]]]]
```

Using this post [14], we assume an approximation of `Table[entropy[v],{v,3,Infinity}]` or $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1,G_v^{\star\star},\omega'),\psi')))$ is:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1,G_v^{\star\star},\omega'),\psi'))) \sim 2\log_2(v)+1-\log_2(3\pi) \tag{51}$$

Hence, using §5.3.2 (a) and §5.3.2 (1), take $|\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')| = \sum_{M=1}^{v}\phi(M) \approx \frac{3}{\pi^2}v^2$ (where ϕ is Euler's Totient function) to compute the following:

$$\begin{aligned}
&\overline{|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)|} = && (52)\\
&\inf\left\{|\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')| : v\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,v},\ \psi'\in\Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')))\geq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)))\right\} = \\
&\inf\left\{\frac{3}{\pi^2}v^2 : r\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,v},\ \psi'\in\Psi_{\varepsilon,v,\omega},\ 2\log_2(v)+1-\log_2(3\pi)\geq\log_2(r!)\right\} =
\end{aligned}$$

where:

(1) For every $r\in\mathbb{N}$, we find a $v\in\mathbb{N}$, where $2\log_2(v)+1-\log_2(3\pi)\geq\log_2(r!)$, but the absolute value of $(2\log_2(v)+1-\log_2(3\pi))-\log_2(r!)$ is minimized. In other words, for every $r\in\mathbb{N}$, we want $v\in\mathbb{N}$ where:

$$2\log_2(v)+1-\log_2(3\pi)\geq\log_2(r!) \tag{53}$$

$$2^{2\log_2(v)}\geq\log_2(r!)-1+\log_2(3\pi) \tag{54}$$

$$\left(2^{\log_2(v)}\right)^2\geq 2^{\log_2(r!)-1+\log_2(3\pi)} \tag{55}$$

$$v^2\geq\left(2^{\log_2(r!)}2^{\log_2(3\pi)}\right)/2 \tag{56}$$

$$v\geq\sqrt{\frac{r!(3\pi)}{2}} \tag{57}$$

$$v=\left\lceil\sqrt{\frac{3\pi r!}{2}}\right\rceil \tag{58}$$

$$\frac{3}{\pi^2}v^2=\frac{3}{\pi^2}\left(\left\lceil\sqrt{\frac{3\pi r!}{2}}\right\rceil\right)^2\sim\overline{|\mathcal{S}(\mathbf{C}(1,G_r^{\star},\omega),\psi)|} \tag{59}$$

Finally, since $|\mathcal{S}(\mathbf{C}(1,G_r^{\star},\omega),\psi)|=r!$, we wish to prove

$$1<\limsup_{\varepsilon\to0}\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\overline{\alpha}(\varepsilon,r,\omega,\psi)<+\infty$$

within §5.3.2 crit. 1:

$$\limsup_{\varepsilon\to0}\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\overline{\alpha}(\varepsilon,r,\omega,\psi)=\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\frac{\overline{|\mathcal{S}(\mathbf{C}(1,G_r^{\star},\omega),\psi)|}}{|\mathcal{S}(\mathbf{C}(1,G_r^{\star},\omega),\psi))|} \tag{60}$$

$$=\lim_{r\to\infty}\frac{\frac{3}{\pi^2}\left(\left\lceil\sqrt{\frac{3\pi r!}{2}}\right\rceil\right)^2}{r!} \tag{61}$$

where using mathematica, we get the limit is greater than one:

Code 4. Limit of eq. 61

```
N[Limit[((3/Pi^2) (Ceiling[Sqrt[(3 Pi r!)/2]])^2)/(r!), r -> Infinity]]

 (*The output is 1.43239*)
```

Also, using §5.3.2 (b) and §5.3.2 (1), take $|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')| = \sum_{M=1}^{v} \phi(M) \approx \frac{3}{\pi^2} v^2$ (where ϕ is Euler's Totient function) computing the following:

$$\begin{aligned}&\underline{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)\right|} = \\ &\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \le \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))\right\} = \\ &\sup\left\{\frac{3}{\pi^2} v^2 : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ 2\log_2(v) + 1 - \log_2(3\pi) \le \log_2(r!)\right\} =\end{aligned} \tag{62}$$

where:

(1) For every $r \in \mathbb{N}$, we find a $v \in \mathbb{N}$, where $2\log_2(r) + 1 - \log_2(3\pi) \le \log_2(r!)$, but the absolute value of $\log_2(r!) - (2\log_2(v) + 1 - \log_2(3\pi))$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $v \in \mathbb{N}$ where:

$$2\log_2(v) + 1 - \log_2(3\pi) \le \log_2(r!) \tag{63}$$

$$2\log_2(v) \le \log_2(r!) - 1 + \log_2(3\pi) \tag{64}$$

$$\left(2^{\log_2(v)}\right)^2 \le 2^{\log_2(r!) - 1 + \log_2(3\pi)} \tag{65}$$

$$(v)^2 \le \left(2^{\log_2(r!)} 2^{\log_2(3\pi)}\right)/2 \tag{66}$$

$$v \le \sqrt{\frac{r!(3\pi)}{2}} \tag{67}$$

$$v = \left\lfloor \sqrt{\frac{3\pi r!}{2}} \right\rfloor \tag{68}$$

$$\frac{3}{\pi^2} v^2 = \frac{3}{\pi^2}\left(\left\lfloor \sqrt{\frac{3\pi r!}{2}} \right\rfloor\right)^2 \sim \underline{|\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi)|} \tag{69}$$

Finally, since $|\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi)| = r!$, we wish to prove

$$1 < \liminf_{\varepsilon \to 0} \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty$$

within §5.3.2 crit. 1:

$$\liminf_{\varepsilon \to 0} \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) = \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \frac{\underline{|\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi)|}}{|\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi))|} \tag{70}$$

$$= \lim_{r \to \infty} \frac{\frac{3}{\pi^2}\left(\left\lfloor \sqrt{\frac{3\pi r!}{2}} \right\rfloor\right)^2}{r!} \tag{71}$$

where using mathematica, we get the limit is greater than one:

Code 5. Limit of eq. 71

```
Clear["*Global`*"]
N[Limit[((3/Pi^2) (Floor[Sqrt[(3 Pi r!)/2]])^2)/(r!), r -> Infinity]]

(*Output is 1.43239*)
```

Hence, since the limits in eq. 61 and eq. 71 are greater than one and less than $+\infty$: i.e.,

$$1 < \liminf_{\varepsilon \to 0} \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) = \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty \tag{72}$$

what we're measuring from $(G_r^{\star})_{r \in \mathbb{N}}$ increases at a rate **superlinear** to that of $(G_v^{\star\star})_{v \in \mathbb{N}}$ (i.e., 5.3.2 crit. 1).

5.3.4. *Example of The "Measure" from* $(G_r^{\star})_{r \in \mathbb{N}}$ *Increasing at a Rate Sub-Linear to that of* $(G_v^{\star\star})_{v \in \mathbb{N}}$. Using our previous example, we can use the following theorem:

Theorem 8. *If what we're measuring from* $(G_r^{\star})_{r \in \mathbb{N}}$ *increases at a rate superlinear to that of* $(G_v^{\star\star})_{v \in \mathbb{N}}$*, then what we're measuring from* $(G_v^{\star\star})_{v \in \mathbb{N}}$ *increases at a rate* ***sublinear*** *to that of* $(G_r^{\star})_{r \in \mathbb{N}}$

Hence, in our definition of super-linear (§5.3.2 crit. 1), swap $G_r^{\star}$ for $G_v^{\star\star}$ *and* $v \in \mathbb{N}$ for $r \in \mathbb{N}$ regarding $\overline{\alpha}\left(\epsilon, r, \omega, \psi\right)$ and $\underline{\alpha}\left(\epsilon, r, \omega, \psi\right)$ (i.e., $\overline{\alpha}\left(\epsilon, v, \omega, \psi\right)$ and $\underline{\alpha}\left(\epsilon, v, \omega, \psi\right)$) and notice thm. 8 is true when:

$$1 < \limsup_{\varepsilon \to 0} \limsup_{v \to \infty} \sup_{\omega \in \Omega_{\varepsilon, v}} \sup_{\psi \in \Psi_{\varepsilon, v, \omega}} \overline{\alpha}\left(\varepsilon, v, \omega, \psi\right), \liminf_{\varepsilon \to 0} \liminf_{v \to \infty} \inf_{\omega \in \Omega_{\varepsilon, v}} \inf_{\psi \in \Psi_{\varepsilon, v, \omega}} \underline{\alpha}\left(\varepsilon, v, \omega, \psi\right) < +\infty$$

5.3.5. *Example of The "Measure" from* $(G_r^{\star})_{r \in \mathbb{N}}$ *Increasing at a Rate Linear to that of* $(G_v^{\star\star})_{v \in \mathbb{N}}$. Suppose, we have function $f : A \to \mathbb{R}$, where $A = \mathbb{Q} \cap [0,1]$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0,1] \end{cases} \tag{73}$$

such that:

$$(A_r^{\star})_{r \in \mathbb{N}} = (\{c/r! : c \in \mathbb{Z}, 0 \leq c \leq r!\})_{r \in \mathbb{N}}$$

and

$$(A_v^{\star\star})_{v \in \mathbb{N}} = (\left\{c/(v!)^2 : c \in \mathbb{N}, 1 \leq c \leq (v!)^2\right\})_{v \in \mathbb{N}}$$

where for $f_r^{\star} : A_r^{\star} \to \mathbb{R}$,

$$f_r^{\star}(x) = f(x) \text{ for all } x \in A_r^{\star} \tag{74}$$

and $f_v^{\star\star} : A_v^{\star\star} \to \mathbb{R}$

$$f_v^{\star\star}(x) = f(x) \text{ for all } x \in A_v^{\star\star} \tag{75}$$

Hence, when $(G_r^{\star})_{r \in \mathbb{N}}$ is:

$$(G_r^{\star})_{r \in \mathbb{N}} = (\{(x, f_r^{\star}(x)) : x \in A_r^{\star}\})_{r \in \mathbb{N}} \tag{76}$$

and $(G_v^{\star\star})_{v \in \mathbb{N}}$ is:

$$(G_v^{\star\star})_{v \in \mathbb{N}} = (\{(x, f_v^{\star\star}(x)) : x \in A_v^{\star\star}\})_{v \in \mathbb{N}} \tag{77}$$

We already know, using eq. 50:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_r^{\star}, \omega), \psi))) \sim \log_2(r! - 2) \sim \log_2(r!) \tag{78}$$

Also, using §5.3.3 steps 1-6 on $(G_v^{\star\star})_{v \in \mathbb{N}}$:

Code 6. Illustration of step (1)-(6) on $(G_v^{\star\star})_{v \in \mathbb{N}}$

```
(*We're using Mathematica*)

Clear["*Global`*"]
A[v_] := A[v] =  Range[0, 7 (v!)]/(7 (v!))

(*Below is step 1*)
S1[v_] :=
 S1[v] = Sort[Select[A[v], Boole[IntegerQ[Denominator[#]/2]] == 1 &]]
S2[v_] :=
 S2[v] = Sort[Select[A[v], Boole[IntegerQ[Denominator[#]/2]] == 0 &]]

(*Below is step 2*)
Dist1[v_] := Dist1[v] = Differences[S1[v]]

(*Below is step 3*)
Dist2[v_] := Dist2[v] = Differences[S2[v]]

(*Below is step 4*)
NonOutliers[v_] :=
 NonOutliers[v] = Dist1[v] (*Dist2[v] is an outlier*)

(*Below is step 5*)
P[v_] := P[v] = NonOutliers[v]/Total[NonOutliers[v]]

(*Below is step 6*)

entropy[v_] := entropy[v] = N[Total[-P[v] Log[2, P[v]]]]

T = Table[{v,entropy[v]},{v,3,6}]
```

where the output is

Code 7. Output of Code 6

```
{{3,(8 Log[17])/(17 Log[2]) + (9 Log[34])/(17 Log[2])},
    {4,(8 Log[287])/(287 Log[2]) + (279 Log[574])/(287 Log[2])},
    {5,(224 Log[7199])/(7199 Log[2]) + (6975 Log[14398])/(7199 Log[2])},
    {6,(2024 Log[259199])/(259199 Log[2]) + (257175 Log[518398])/(259199 Log[2])}}
```

Notice when:

1. $c(v) = (v!)^2/2 - 1$
2. $\{b(4) \mapsto 9,\ b(5) \mapsto 279,\ b(6) \mapsto 6975,\ b(7) \mapsto 257175,\ b(8) \mapsto 19845\}$
3. $a(v) + b(v) = c(v)$

the output of code 7 can be defined:

$$\frac{a(v)\log_2(c(v))}{c(v)} + \frac{b(v)\log(2c(v))}{c(v)} = \frac{a(v)\log_2(c(v)) + b(v)\log(2c(v))}{c(v)} \tag{79}$$

Hence, since $a(v) = c(v) - b(v) = (v!)^2/2 - 1 - b(v)$:

$$\frac{a(v)\log_2(c(v)) + b(v)\log(2c(v))}{c(v)} = \tag{80}$$

$$\frac{((v!)^2/2 - 1 - b(v))\log_2(c(v)) + b(v)\log_2(2c(v))}{c(v)} = \tag{81}$$

$$\frac{((v!)^2/2)\log_2(c(v)) - \log_2(c(v)) - b(v)\log_2(v) + b(v)\log_2(c(v)) + b(v)\log_2(2)}{c(v)} = \tag{82}$$

$$\frac{((v!)^2/2)\log_2(c(v)) - \log_2(c(v)) + b(v)}{c(v)} = \tag{83}$$

$$\frac{((v!)^2/2 - 1)\log_2(c(v)) + b(v)}{c(v)} = \tag{84}$$

$$\frac{((v!)^2/2 - 1)\log_2((v!)^2/2 - 1) + b(v)}{(v!)^2/2 - 1} = \tag{85}$$

$$\log_2((v!)^2/2 - 1) + \frac{b(v)}{(v!)^2/2 - 1} = \tag{86}$$

since $\lim_{v\to\infty} b(v)/c(v) = 1$ (this is proven in [15]):

$$\log_2((v!)^2/2 - 1) + \frac{b(v)}{(v!)^2/2 - 1} \sim \log_2((v!)^2/2 - 1) + 1 \tag{87}$$

$$\log_2((v!)^2/2 - 1) + \log_2(2) = \tag{88}$$

$$\log_2((v!)^2 - 2)) \sim \tag{89}$$

$$\log_2((v!)^2) = \tag{90}$$

$$2\log_2(v!) \tag{91}$$

Hence, `entropy[r]` is the same as:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(1, G_v^{\star\star}, \omega), \psi))) \sim \tag{92}$$

$$2\log_2(v!) \tag{93}$$

Therefore, using §5.3.2 (b) and §5.3.2 (3a), take $|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')| = (v!)^2$ to compute the following:

$$\underline{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)\right|} = \tag{94}$$

$$\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \leq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))\right\} =$$

$$\sup\left\{(v!)^2 : r \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ 2\log_2(v!) \leq \log_2(r!)\right\} =$$

where:

(1) For every $r \in \mathbb{N}$, we find a $v \in \mathbb{N}$, where $2\log_2(v!) \leq \log_2(r!)$, but the absolute value of $\log_2(r!) - 2\log_2(v!)$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $v \in \mathbb{N}$ where:

$$2\log_2(v!) \leq \log_2(r!) \tag{95}$$

$$2^{2\log_2(v!)} \leq 2^{\log_2(r!)} \tag{96}$$

$$(2^{\log_2(v!)})^2 \leq r! \tag{97}$$

$$(v!)^2 \leq r! \tag{98}$$

$$(v!)^2 = \lfloor r! \rfloor \tag{99}$$

To solve for v, we try the following code:

Code 8. Code for v in eq. 99

```
(*We're using Mathematica*)

Clear[”Global‘*”]

T1 = Table[
   {sol[r_] := sol[r] = Reduce[v > 0 && ((v!)^2) <= r!, v, Integers],
    vsolve = Max[v /. Solve[sol[r], {v}, Integers]],
    (* Largest v that solves inequality (v!)^2<=r for every r *)
    , N[(vsolve!)^2/(r!)]}, {r, 3, 40}];

Tablevsolve =
 Table[{T1[[r - 3 + 1, 2]], r}, {r, 3,
   40}] (*Takes largest v-values for every r in r!*)

loweralphr =
 Table[{r, T1[[r - 3 + 1, 4]]}, {r, 3,
   40}] (* Takes largest largest v-values and corresponding r value*)

ListPlot[loweralphr] (*Graph points of loweralphr. Notice, the graph has
a lower bound of zero.*)
```

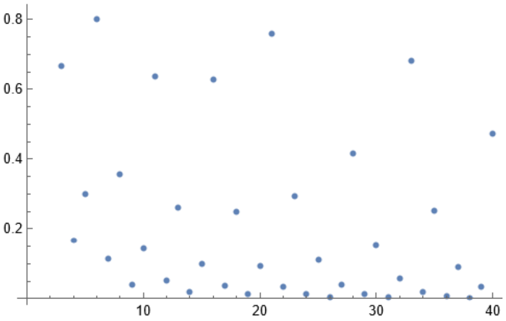
Note, the output is:

Code 9. Output for code 8

```
Clear[”Global‘*”]
(* Output of Tablevsolve*)
{{2, 3}, {2, 4}, {3, 5}, {4, 6}, {4, 7}, {5, 8}, {5, 9}, {6, 10}, {7, 11}, {7, 12},
 {8, 13}, {8, 14}, {9, 15}, {10, 16}, {10, 17}, {11, 18}, {11, 19}, {12, 20}, {13, 21},
 {13, 22}, {14, 23}, {14, 24}, {15, 25}, {15, 26}, {16, 27}, {17, 28}, {17, 29},
 {18, 30}, {18, 31}, {19, 32}, {20, 33}, {20, 34}, {21, 35}, {21, 36}, {22, 37}, {22, 38},
 {23, 39}, {24, 40}}

(*Output of loweralphr*)
{{3, 0.666667}, {4, 0.166667}, {5, 0.3}, {6, 0.8}, {7, 0.114286}, {8, 0.357143}, {9, 0.0396825},
 {10, 0.142857}, {11, 0.636364}, {12, 0.0530303}, {13, 0.261072}, {14, 0.018648}, {15, 0.100699},
 {16, 0.629371}, {17, 0.0370218}, {18, 0.248869}, {19, 0.0130984}, {20, 0.0943082}, {21, 0.758956},
 {22, 0.034498}, {23, 0.293983}, {24, 0.0122493}, {25, 0.110244}, {26, 0.00424014}, {27,0.0402028},
 {28, 0.41495}, {29, 0.0143086}, {30, 0.154533}, {31, 0.00498494}, {32, 0.0562364}, {33, 0.681653},
 {34, 0.0200486}, {35, 0.252613}, {36, 0.00701702}, {37, 0.0917902}, {38, 0.00241553},
 {39, 0.0327645}, {40, 0.471809}}
```

FIGURE 1. Plot of `loweralphr`



Finally, since the lower bound of `loweralphr` is zero, we have shown:

$$\liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}\left(\varepsilon,r,\omega,\psi\right)=0 \tag{100}$$

Next, using §5.3.2 (b) and §5.3.2 (3b), take $|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega'),\psi')|=r!$ and swap $r\in\mathbb{N}$ and $(G_r^{\star})_{r\in\mathbb{N}}$ with $v\in\mathbb{N}$ and $(G_v^{\star\star})_{v\in\mathbb{N}}$, to compute the following:

$$\begin{aligned}
&\underline{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega),\psi)\right|}= \\
&\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega'),\psi')\right|:r\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,r},\ \psi'\in\Psi_{\varepsilon,r,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega'),\psi')))\geq\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega),\psi)))\right\}= \\
&\inf\left\{r!:r\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,r},\ \psi'\in\Psi_{\varepsilon,r,\omega},\ \log_2(r!)\geq 2\log_2(v!)\right\}=
\end{aligned} \tag{101}$$

where:

(1) For every $v\in\mathbb{N}$, we find a $r\in\mathbb{N}$, where $\log_2(r!)\leq 2\log_2(v!)$, but the absolute value of $2\log_2(v!)-\log_2(r!)$ is minimized. In other words, for every $v\in\mathbb{N}$, we want $r\in\mathbb{N}$ where:

$$\log_2(r!)\leq 2\log_2(v!) \tag{102}$$

$$2^{\log_2(r!)}\leq 2^{2\log_2(v!)} \tag{103}$$

$$r!\leq(2^{\log_2(v!)})^2 \tag{104}$$

$$r!\leq(v!)^2 \tag{105}$$

$$r!=(v!)^2 \tag{106}$$

To solve r, we try the following code:

CODE 10. Code for r in eq. 106

```
(*We're using Mathematica*)

Clear["Global`*"]
T2 = Table[
   {sol[v_] := sol[v] = Reduce[v > 0 && r! <= (v!)^2, r, Integers],
    rsolve = Max[r /. Solve[sol[v], {r}, Integers]],
    (* Largest r that solves inequality (r!)<=(v!)^2 for every v *)
    , N[(rsolve!)/((v!)^2)]}, {v, 3, 40}];

Tablersolve =
 Table[{T2[[v - 3 + 1, 2]], v}, {v, 3,
   40}] (*Takes largest r-values for every v in (v!)^2*)

loweralphv =
 Table[{v, T2[[v - 3 + 1, 4]]}, {v, 3,
   40}] (* Takes largest largest r values and corresponding v value*)

ListPlot[loweralphv](*Graph points of loweralphv. Notice, the graph
has a lower bound of zero*)
```

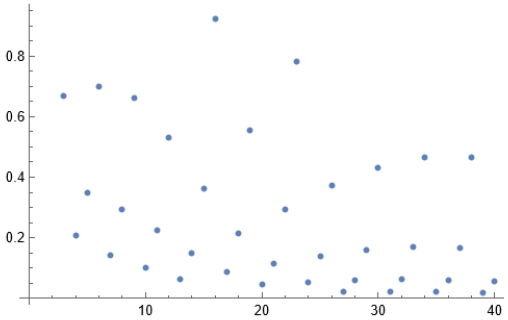
Note, the output is:

CODE 11. Output for code 10

```
Clear["Global`*"]
(* Output of Tablersolve*)
{{4, 3}, {5, 4}, {7, 5}, {9, 6}, {10, 7}, {12, 8}, {14, 9}, {15, 10}, {17, 11}, {19, 12},
 {20, 13}, {22, 14}, {24, 15}, {26, 16}, {27, 17}, {29, 18}, {31, 19}, {32, 20}, {34, 21},
 {36, 22}, {38, 23}, {39, 24}, {41, 25}, {43, 26}, {44, 27}, {46, 28}, {48, 29}, {50, 30},
 {51, 31}, {53, 32}, {55, 33}, {57, 34}, {58, 35}, {60, 36}, {62, 37}, {64, 38}, {65, 39},
 {67, 40}}

(*Output of loweralphv*)
{{3, 0.666667}, {4, 0.166667}, {5, 0.3}, {6, 0.8}, {7, 0.114286}, {8, 0.357143}, {9, 0.0396825},
 {10, 0.142857}, {11, 0.636364}, {12, 0.0530303}, {13, 0.261072}, {14, 0.018648}, {15, 0.100699},
 {16, 0.629371}, {17, 0.0370218}, {18, 0.248869}, {19, 0.0130984}, {20, 0.0943082}, {21, 0.758956},
 {22, 0.034498}, {23, 0.293983}, {24, 0.0122493}, {25, 0.110244}, {26, 0.00424014}, {27,0.0402028},
 {28, 0.41495}, {29, 0.0143086}, {30, 0.154533}, {31, 0.00498494}, {32, 0.0562364}, {33, 0.681653},
 {34, 0.0200486}, {35, 0.252613}, {36, 0.00701702}, {37, 0.0917902}, {38, 0.00241553},
 {39, 0.0327645}, {40, 0.471809}}
```

FIGURE 2. Plot of `loweralphv`



since the lower bound of `loweralphv` is zero, we have shown:

$$\liminf_{\varepsilon\to 0}\liminf_{v\to\infty}\inf_{\omega\in\Omega_{\varepsilon,v}}\inf_{\psi\in\Psi_{\varepsilon,v,\omega}}\underline{\alpha}\left(\varepsilon,v,\omega,\psi\right)=0 \tag{107}$$

Hence, using eq. 100 and 107, since **both**:

(1) $\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\overline{\alpha}\left(\varepsilon,r,\omega,\psi\right)$ or $\liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}\left(\varepsilon,r,\omega,\psi\right)$ are equal to zero, one or $+\infty$

(2) $\limsup_{\varepsilon\to 0}\limsup_{v\to\infty}\sup_{\omega\in\Omega_{\varepsilon,v}}\sup_{\psi\in\Psi_{\varepsilon,v,\omega}}\overline{\alpha}\left(\varepsilon,v,\omega,\psi\right)$ or $\liminf_{\varepsilon\to 0}\liminf_{v\to\infty}\inf_{\omega\in\Omega_{\varepsilon,v}}\inf_{\psi\in\Psi_{\varepsilon,v,\omega}}\underline{\alpha}\left(\varepsilon,v,\omega,\psi\right)$ are equal to zero, one or $+\infty$

then *what I'm measuring from* $(G_r^{\star})_{r\in\mathbb{N}}$ increases at a rate **linear** to that of $(G_v^{\star\star})_{v\in\mathbb{N}}$.

5.4. **Defining The Actual Rate of Expansion of a Sequence of Bounded Sets.**

5.4.1. *Definition of Actual Rate of Expansion of a Sequence of Bounded Sets.* Suppose:

(1) $(G_r^{\star})_{r\in\mathbb{N}}$ is a sequence of the graph of each $f_r^{\star}$ (§3.1C)
(2) C is a reference point in $\mathbb{R}^{n+1}$
(3) $Q, R\in\mathbb{R}^{n+1}$
(4) $Q=(q_1,\cdots,q_{n+1})$ and $R=(r_1,\cdots,r_{n+1})$, where:

$$Q-R=(q_1-r_1,\cdots,q_{n+1}-r_{n+1})$$

(5) $||Q||_{n+1}=\sqrt{q_1^2+\cdots+q_{n+1}^2}$ and $||R||_{n+1}=\sqrt{r_1^2+\cdots+r_{n+1}^2}$
(6) $C-G_r^{\star}=\{C-y:y\in G_r^{\star}\}$
(7) $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension
(8) $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra

For any $r \in \mathbb{N}$, take the $(n+1)$-dimensional Euclidean distance between a reference point $C \in \mathbb{R}^{n+1}$ and each point in $G_r^{\star}$:

$$\mathcal{G}(C, G_r^{\star}) = \{||C - y|| : y \in G_r^{\star}\}$$

then average $\mathcal{G}(C, G_r^{\star})$:

$$\text{Avg}(\mathcal{G}(C, G_r^{\star})) = \frac{1}{\mathcal{H}^{\dim_{\text{H}}(C - G_r^{\star})}(C - G_r^{\star})} \int_{C - G_r^{\star}} ||(x_1, \cdots, x_{n+1})||_{n+1} \, d\mathcal{H}^{\dim_{\text{H}}(C - G_r^{\star})}$$

where **the actual rate of expansion of** $(G_r^{\star})_{r \in \mathbb{N}}$ is:

$$\mathcal{E}(C, G_r^{\star}) = \text{Avg}(\mathcal{G}(C, G_{r+1}^{\star})) - \text{Avg}(\mathcal{G}(C, G_r^{\star}))$$

If $\mathcal{E}(C, G_r^{\star})$ is undefined, replace the Hausdorff measure $\mathcal{H}^{\dim_{\text{H}}(C - G_r^{\star})}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi_{h,g}^{\mu}(q,t)}$ [1, p.26-33]

5.4.2. *Example.* Suppose, we have $f : A \to \mathbb{R}$, where $A = \mathbb{R}$ and $f(x) = x$, such that $(A_r^{\star})_{r \in \mathbb{N}} = ([-r, r])_{r \in \mathbb{N}}$ and for $f_r^{\star} : A_r^{\star} \to \mathbb{R}$:

$$f_r^{\star}(x) = f(x) \text{ for all } x \in A_r^{\star}$$

Hence, when $(G_r^{\star})_{r \in \mathbb{N}}$ is:

$$(G_r^{\star})_{r \in \mathbb{N}} = (\{(x, x) : x \in [-r, r]\})_{r \in \mathbb{N}}$$

such that $C = (0, 0)$, note:

$$\text{Avg}(\mathcal{G}(C, G_r^{\star})) = \frac{1}{\mathcal{H}^{\dim_{\text{H}}(C - G_r^{\star})}(C - G_r^{\star})} \int_{C - G_r^{\star}} ||(x_1, x_2)||_2 \, d\mathcal{H}^{\dim_{\text{H}}(C - G_r^{\star})} = \tag{108}$$

$$\frac{1}{\mathcal{H}^1((0,0) - G_r^{\star})} \int_{-r}^{r} ||(x_1, x_1)||_2 \, d\mathcal{H}^1 = \ (\text{since } x_2 = f(x_1) = x_1) \tag{109}$$

$$\frac{1}{\text{length}(G_r^{\star})} \int_{-r}^{r} \sqrt{x_1^2 + x_1^2} \, dx_1 = \tag{110}$$

$$\frac{1}{\sqrt{(r - (-r))^2 - (r - (-r))^2}} \int_{-r}^{r} \sqrt{2x_1^2} \, dx_1 = \tag{111}$$

$$\frac{1}{\sqrt{(2r)^2 + (2r)^2}} \int_{-r}^{r} \sqrt{2x_1^2} \, dx_1 = \tag{112}$$

$$\frac{1}{2\sqrt{2}r} \int_{-r}^{r} \sqrt{2}|x_1| \, dx_1 = \tag{113}$$

$$\frac{1}{2\sqrt{2}r} \left(\frac{\sqrt{2}}{2} \text{sign}(x_1)(x_1)^2 \Big|_{-r}^{r} \right) = \tag{114}$$

$$\frac{1}{2\sqrt{2}r} \left(\frac{\sqrt{2}}{2} \text{sign}(r) r^2 - \frac{\sqrt{2}}{2} \text{sign}(-r)(-r)^2 \right) = \tag{115}$$

$$\frac{1}{2\sqrt{2}r} \left(\frac{\sqrt{2}}{2} r^2 + \frac{\sqrt{2}}{2} r^2 \right) = \tag{116}$$

$$\frac{1}{2\sqrt{2}r} \left(\sqrt{2} r^2 \right) = \tag{117}$$

$$\frac{1}{2} r \tag{118}$$

$$\tag{119}$$

and the actual rate of expansion is:

$$\begin{aligned}\mathcal{E}(C,G_r^{\star}) =& \operatorname{Avg}(\mathcal{G}(C,G_{r+1}^{\star}))-\operatorname{Avg}(\mathcal{G}(C,G_r^{\star})) = && (120)\\ & (r+1)/2-r/2= && (121)\\ & (1/2)r+1/2-(1/2)r= && (122)\\ & 1/2 && (123)\end{aligned}$$

5.5. **Reminder.** See if §3.1 is easier to understand.

6. My Attempt At Answering The Approach of §2.5.1

6.1. **Choice Function.** Suppose we define the following:

1. If for all $r\in\mathbb{N}$, $\mathcal{B}\subset\mathbf{B}(\mathbb{R}^n)$ (§2.3.3) is an arbitrary set, where $A_r^{\star}\in\mathcal{B}$ and $\mathscr{B}\subset\mathfrak{B}(A_r^{\star})$ (§2.3.3) is an arbitrary set, then $f_r^{\star}\in\mathscr{B}$ satisfies (1), (2), (3), (4) and (5) of the **leading question** in §3.1
2. For all $v\in\mathbb{N}$, $A_v^{\star\star}\in\mathbf{B}(\mathbb{R}^n)\setminus\mathcal{B}$ and $f_v^{\star\star}\in\mathfrak{B}(A_v^{\star\star})\cup(\mathfrak{B}(A_r^{\star})\setminus\mathscr{B})$

Further note, from §5.3.2 (a), if we take:

$$\begin{aligned}&\overline{|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)|}= && (124)\\ &\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')\right| : v\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,v},\ \psi'\in\Psi_{\varepsilon,v,\omega},\ \operatorname{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')))\geq\operatorname{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)))\right\}\end{aligned}$$

and from §5.3.2 (b), we take:

$$\begin{aligned}&\underline{|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)|}= && (125)\\ &\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')\right| : v\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,v},\ \psi'\in\Psi_{\varepsilon,v,\omega},\ \operatorname{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_v^{\star\star},\omega'),\psi')))\leq\operatorname{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)))\right\}\end{aligned}$$

Then, §5.3.1 (2), eq. 124, and eq. 125 is:

$$\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)|=|\mathcal{S}'(\varepsilon,G_r^{\star})|=|\mathcal{S}'| \tag{126}$$

$$\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\overline{|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)|}=\overline{|\mathcal{S}'(\varepsilon,G_r^{\star})|}=\overline{|\mathcal{S}'|} \tag{127}$$

$$\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{|\mathcal{S}(\mathbf{C}(\varepsilon,G_r^{\star},\omega),\psi)|}=\underline{|\mathcal{S}'(\varepsilon,G_r^{\star})|}=\underline{|\mathcal{S}'|} \tag{128}$$

6.2. **Approach.** We manipulate the definitions of §5.3.2 (a) and §5.3.2 (b) to solve (1), (2), (3), (4) and (5) of the *leading question* in §3.1

6.3. Potential Answer.

6.3.1. *Preliminaries (Definition of T).* Suppose $(G_r^{\star})_{r\in\mathbb{N}}$ is the sequence of the graph on each function $f_r^{\star}$ (§2.3.1). Then, when:

- The average of $G_r^{\star}$ for every $r\in\mathbb{N}$ is:

$$\operatorname{Avg}(G_r^{\star})=\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(G_r^{\star})}(G_r^{\star})}\int_{G_r^{\star}}(x_1,\cdots,x_{n+1})\,d\mathcal{H}^{\dim_{\mathrm{H}}(G_r^{\star})} \tag{129}$$

- $d(P,Q)$ is the $(n+1)$-dimensional Euclidean distance between points $P,Q\in\mathbb{R}^{n+1}$
- The difference of point $X=(x_1,\cdots,x_{n+1})$ and $Y=(y_1,\cdots,y_{n+1})$ is:

$$X-Y=(x_1-y_1,x_2-y_2,\cdots,x_{n+1}-y_{n+1})$$

We define an *explicit* injective $\mathscr{F}:\mathbb{R}^n\to\mathbb{R}$, where $r,v\in\mathbb{N}$, such that:

1. If $d(\operatorname{Avg}(G_r^{\star}),C)<d(\operatorname{Avg}(G_v^{\star\star}),C)$, then $\mathscr{F}(\operatorname{Avg}(G_r^{\star})-C)<\mathscr{F}(\operatorname{Avg}(G_v^{\star\star})-C)$
2. If $d(\operatorname{Avg}(G_r^{\star}),C)>d(\operatorname{Avg}(G_v^{\star\star}),C)$, then $\mathscr{F}(\operatorname{Avg}(G_r^{\star})-C)>\mathscr{F}(\operatorname{Avg}(G_v^{\star\star})-C)$
3. If $d(\operatorname{Avg}(G_r^{\star}),C)=d(\operatorname{Avg}(G_v^{\star\star}),C)$, then $\mathscr{F}(\operatorname{Avg}(G_r^{\star})-C)\neq\mathscr{F}(\operatorname{Avg}(G_v^{\star\star})-C)$

where we define:

$$T(C,G_r^{\star})=\mathscr{F}(\operatorname{Avg}(G_r^{\star})-C) \tag{130}$$

6.3.2. *Question.* Does T exist? If so, how do we define it?
Hence, using $|\mathcal{S}'|$, $\overline{|\mathcal{S}'|}$, $\underline{|\mathcal{S}'|}$, E, $\mathcal{E}(C, G_r^{\star})$ (§5.4), and $T(C, G_r^{\star})$, such that with the absolute value function $||\cdot||$, ceiling function $\lceil\cdot\rceil$, and nearest integer function $[\cdot]$, we define:

$$\begin{aligned}&K(\varepsilon, G_r^{\star}) = \\ &(1 + ||E - \mathcal{E}(C, G_r^{\star})||)\left(\left|\left|\frac{|\mathcal{S}'|\left(1 + \left\lceil\frac{|\mathcal{S}'|\left(\underline{|\mathcal{S}'|}+2|\mathcal{S}'|\right)}{\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|\right)\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|+\overline{|\mathcal{S}'|}\right)}\right\rceil\right)\left(1 + \left[\underline{|\mathcal{S}'|}/\,|\mathcal{S}'|\right]\right)}{\left(1 + \left[|\mathcal{S}'|\,/\overline{|\mathcal{S}'|}\right]\right)\left(1 + \left[\underline{|\mathcal{S}'|}/\overline{|\mathcal{S}'|}\right]\right)} - \left|\mathcal{S}'\right|\right|\right| + \left|\mathcal{S}'\right|\right) - T(C, G_r^{\star})\mathcal{E}(C, G_r^{\star}) \qquad (131)\end{aligned}$$

where $\mathcal{E}$, E, and T are "removed" when $\mathcal{E}, E = 0$, the choice function which answers the **leading question** in §3.1 could be the following, s.t. we explain the reason behind choosing the choice function in §6.4:

Theorem 9. *If we define:*

$$\mathcal{M}(\varepsilon, G_r^{\star}) = |\mathcal{S}'(\varepsilon, G_r^{\star})|\big(K(\varepsilon, G_r^{\star}) - |\mathcal{S}'(\varepsilon, G_r^{\star})|\big)$$

$$\mathcal{M}(\varepsilon, G_v^{\star\star}) = |\mathcal{S}'(\varepsilon, G_v^{\star\star})|\big(K(\varepsilon, G_v^{\star\star}) - |\mathcal{S}'(\varepsilon, G_v^{\star\star})|\big)$$

where for $\mathcal{M}(\varepsilon, G_r^{\star})$, we define $\mathcal{M}(\varepsilon, G_r^{\star})$ to be the same as $\mathcal{M}(\varepsilon, G_v^{\star\star})$ when swapping "$v \in \mathbb{N}$" with "$r \in \mathbb{N}$" (for eq. 124 *&* 125*) and sets $G_r^{\star}$ with $G_v^{\star\star}$ (for eq. 124–131), then for constant $v > 0$ and variable $v^* > 0$, if:*

$$\overline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star}) = \inf\left(\{|\mathcal{S}'(\varepsilon, G_v^{\star\star})| : v \in \mathbb{N}, \mathcal{M}(\varepsilon, G_v^{\star\star}) \geq \mathcal{M}(\varepsilon, G_r^{\star}) \geq v^*\} \cup \{v^*\}\right) + v \qquad (132)$$

and:

$$\underline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star}) = \sup\left(\{|\mathcal{S}'(\varepsilon, G_v^{\star\star})| : v \in \mathbb{N}, v^* \leq \mathcal{M}(\varepsilon, G_v^{\star\star}) \leq \mathcal{M}(\varepsilon, G_r^{\star})\} \cup \{-v^*\}\right) + v \qquad (133)$$

where for all $r, v \in \mathbb{N}$, there exists a $A_r^{\star} \in \mathcal{B}$ and $f_r^{\star} \in \mathscr{B}$ (§6.1 crit. 1*), such that for all $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$ and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star}) \cup (\mathfrak{B}(A_r^{\star}) \setminus \mathscr{B})$ (§6.1 crit.* 2*), whenever:*

$$\inf\left\{||1 - c|| : \forall(\epsilon > 0)\exists(c > 0)\forall(r \in \mathbb{N})\exists(v \in \mathbb{N})\left(\left|\left|\frac{|\mathcal{S}'(\varepsilon, G_r^{\star})|}{|\mathcal{S}'(\varepsilon, G_v^{\star\star})|} - c\right|\right| < \varepsilon\right)\right\} \qquad (134)$$

such that $\lceil\cdot\rceil$ is the ceiling function, E is the fixed rate of expansion, Γ is the gamma function, n is the dimension of $\mathbb{R}^n$, $dim_H(G_r^{\star})$ is the Hausdorff dimension of set $G_r^{\star} \subseteq \mathbb{R}^{n+1}$, and $\mathbf{A}_r$ is area of the smallest $(n+1)$-dimensional box that contains $A_r^{\star}$, then:

$$\begin{aligned}&V(\varepsilon, G_r^{\star} n) = \qquad (135)\\ &\left\lceil\left(\mathbf{A_r}^{1-\operatorname{sign}(E)}(E - \operatorname{sign}(E) + 1)\left(\frac{\exp\left(n\ln(\pi)/2\right)}{\Gamma(n/2+1)}\right)\left(r!^{(n-\dim_{\mathrm{H}}(G_r^{\star}))}\right)\left(r^{\operatorname{sign}(E)\left(\dim_{\mathrm{H}}(G_r^{\star})-\operatorname{sign}\left(\dim_{\mathrm{H}}(G_r^{\star})\right)+1\right)}\right) + \right.\right.\\ &\left.\left.(1 - \operatorname{sign}(\dim_{\mathrm{H}}(G_r^{\star})))\right)/\varepsilon\right\rceil \Big/ |\mathcal{S}'(\varepsilon, G_r^{\star})|\end{aligned}$$

& the choice function is:

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\lim_{v^*\to\infty}\limsup_{r\to\infty}\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^{\star}))\overline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^{\star})| + v} - c^{-V(\varepsilon, G_r^{\star} n)}\right) \qquad (136)\\ &\qquad\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^{\star}))\underline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^{\star})| + v} - c^{-V(\varepsilon, G_r^{\star} n)}\right) = \\ &\liminf_{\varepsilon\to 0}\lim_{v^*\to\infty}\liminf_{r\to\infty}\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^{\star}))\overline{\mathcal{S}}(\varepsilon, k, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^{\star})| + v} - c^{-V(\varepsilon, G_r^{\star} n)}\right) \qquad (137)\\ &\qquad\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^{\star}))\underline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^{\star})| + v} - c^{-V(\varepsilon, G_r^{\star} n)}\right) = 0\end{aligned}$$

where $(G_r^{\star})_{r\in\mathbb{N}}$ satisfies eq. 136 *& eq.* 137*. (Note, we want $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$) such that the expected value which answers the approach of* §2.5.1*, using the leading question* (§3.1)*, is* $\mathbb{E}[f_r^{\star}]$

6.4. **Explaining The Choice Function and Evidence The Choice Function Is Credible.** Notice, before reading the programming in code 12, without the "c"-terms in eq. 136 and eq. 137:

(1) The choice function in eq. 136 and eq. 137 is zero, when *what I'm measuring* from $(G_r^\star)_{r\in\mathbb{N}}$ (§5.3.2 criteria 1) increases at a rate superlinear to that of $(G_v^{\star\star})_{v\in\mathbb{N}}$, where $\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star)) = 0$.

(2) The choice function in eq. 136 and eq. 137 is zero, when for a given $(G_r^\star)_{r\in\mathbb{N}}$ and $(G_v^{\star\star})_{v\in\mathbb{N}}$ there doesn't exist c where eq. 134 is satisfied or $c = 0$.

(3) When c does exist, suppose:

$$\left\{\mathcal{J}(r) : r \in \mathbb{N}, \frac{|\mathcal{S}'(\varepsilon, G_r^\star)|}{|\mathcal{S}'(\varepsilon, G_{\mathcal{J}(r)}^{\star\star})|} \approx c\right\} \tag{138}$$

   (a) When $|\mathcal{S}'(\varepsilon, G_r^\star)| < |\mathcal{S}'(\varepsilon, G_{\mathcal{J}(r)}^{\star\star})|$, then:

$$\limsup_{\varepsilon\to 0} \lim_{v^*\to\infty} \limsup_{r\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\overline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} = c \tag{139}$$

$$\liminf_{\varepsilon\to 0} \lim_{v^*\to\infty} \liminf_{r\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\underline{\mathcal{S}}(\varepsilon, r, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} = 0 \tag{140}$$

   (b) When $|\mathcal{S}'(\varepsilon, G_r^\star)| > |\mathcal{S}'(\varepsilon, G_{\mathcal{J}(r)}^{\star\star})|$, then:

$$\limsup_{\varepsilon\to 0} \lim_{v^*\to\infty} \limsup_{r\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\overline{\mathcal{S}}(\varepsilon, k, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} = +\infty \tag{141}$$

$$\liminf_{\varepsilon\to 0} \lim_{v^*\to\infty} \liminf_{r\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))\underline{\mathcal{S}}(\varepsilon, k, v^*, G_v^{\star\star})}{|\mathcal{S}'(\varepsilon, G_r^\star)| + v} = 1/c \tag{142}$$

Hence, for each sub-criteria under crit. (3), if we subtract one of their limits by their limit value, then eq. 136 and eq. 137 is zero. (We do this using the "c"-term in eq. 136 and 137). However, when the exponents of the "c"-terms aren't equal to -1, the limits of eq. 136 and 137 aren't equal to zero. We want this, infact, whenever we swap $\mathcal{S}'(\varepsilon, G_r^\star)$ with $\mathcal{S}'(\varepsilon, G_v^{\star\star})$. Moreover, we define function $V(\varepsilon, G_r^\star, n)$ (i.e., eq. 135), where:

   (i) When $S'(\varepsilon, G_r^\star) \gg \text{Numerator}\,(V(\varepsilon, G_r^\star, n))$, then eq. 136 and 137 without the "c"-terms are zero. (The "c"-terms approach zero and still allow eq. 136 and 137 to equal zero.)

   (ii) When $S'(\varepsilon, G_r^\star) \ll \text{Numerator}\,(V(\varepsilon, G_r^\star, n))$, then $\operatorname{sign}(\mathcal{M}(\varepsilon, G_r^\star))$ is zero which makes eq. 136 and 137 equal zero.

   (iii) Here are some examples of the numerator of $V(\varepsilon, G_r^\star, n)$ (eq. 135):

      (A) When $E = 0$, $n = 1$, and $\dim_H(A) = 0$, the numerator of $V(\varepsilon, G_r^\star, n)$ is $\lceil (\mathbf{A}r! + 1)/\varepsilon \rceil$

      (B) When $E = z$, $n = 1$, and $\dim_H(A) = 0$, the numerator of $V(\varepsilon, G_r^\star, n)$ is $\lceil (2zr \cdot r! + 1)/\varepsilon \rceil$

      (C) When $E = 0$, $n = z_2$, and $\dim_H(A) = z_2$, the numerator of $V(\varepsilon, G_r^\star, n)$ is ceiling of constant $\mathbf{A}$ times the volume of an n-dimensional ball with finite radius: i.e.,

$$\left\lceil \frac{\mathbf{A}z_1 \exp\left(z_2 \ln(\pi)/2\right)}{\Gamma(z_2/2 + 1)} \Big/ \varepsilon \right\rceil$$

      (D) When $E = z_1$, $n = z_2$, and $\dim_H(A) = z_2$, the numerator of $V(\varepsilon, G_r^\star, n)$ is ceiling of the volume of the n-dimensional ball: i.e.,

$$\left\lceil \frac{z_1 \exp\left(z_2 \ln(\pi)/2\right)}{\Gamma(z_2/2 + 1)} r^{z_2} \Big/ \varepsilon \right\rceil$$

Now, consider the code for eq. 136 and eq. 137. (Note, the set theoretic limit of $G_r^\star$ is the graph of function $f : A \to \mathbb{R}$.) In this example, $A = \mathbb{Q} \cap [0, 1]$, and:

$$f(x) = \begin{cases} 1 & x \in \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0, 1] \\ 0 & x \notin \{(2s+1)/(2t) : s \in \mathbb{Z}, t \in \mathbb{N}, t \neq 0\} \cap [0, 1] \end{cases} \tag{143}$$

such that:

$$(A_r^\star)_{r\in\mathbb{N}} = (\{c/(r!) : c \in \mathbb{Z}, 0 \leq c \leq r!\})_{r\in\mathbb{N}}$$

the ceiling function is $\lceil \cdot \rceil$, and:

$$(A_v^{\star\star})_{v\in\mathbb{N}} = (\{c/\lceil v!/3 \rceil : c \in \mathbb{Z}, 0 \leq c \leq \lceil v!/3 \rceil\})_{v\in\mathbb{N}}$$

such for $f_r^\star : A_r^\star \to \mathbb{R}$,

$$f_r^\star(x) = f(x) \text{ for all } x \in A_r^\star \tag{144}$$

and $f_v^{\star\star} : A_v^{\star\star} \to \mathbb{R}$

$$f_v^{\star\star}(x) = f(x) \text{ for all } x \in A_v^{\star\star} \tag{145}$$

Hence, when $(G_r^\star)_{r\in\mathbb{N}}$ is:

$$(G_r^\star)_{r\in\mathbb{N}} = (\{(x, f(x)) : x \in \{c/k! : c \in \mathbb{Z}, 0 \le c \le k!\}\})_{r\in\mathbb{N}} \tag{146}$$

and $(G_v^{\star\star})_{v\in\mathbb{N}}$ is:

$$(G_v^{\star\star})_{v\in\mathbb{N}} = (\{(x, f(x)) : x \in \{c/\lceil v!/3\rceil : c \in \mathbb{Z}, 0 \le c \le \lceil v!/3\rceil\}\})_{v\in\mathbb{N}} \tag{147}$$

Note, the following (we leave this to mathematicians to figure `LengthS1`, `LengthS2`, `Entropy1` and `Entropy2` for other A and f in code 12).

Code 12. Code for eq. 136 and 137 to eq. 146 and eq. 147

```
Clear["Global‘*"]

(*‘A’ is the domain of f*)
A=Intersect[Rationals,Inteval[{0,1}]]

(*‘f’ is the function we are averaging over. In the case of f in §5.3.3, this can be
reperesented in mathematica using the following*)
f[x_] := f[x] =
  Piecewise[{{1, Boole[IntegerQ[Denominator[x]/2]] == 1}, {0,
     Boole[IntegerQ[(Denominator[x] - 1)/2]] == 1}}]

eps=1 (*Since 'A' is rational, we set 'eps' or $\varepsilon$ to 1*)

(*‘LengthS1’ is $|S'(\varepsilon, G_r^\star)|$*)
LengthS1[r_] := LengthS1[r] = Ceiling[r!/3] + 1

(*‘Entropy1’ is the approximation of $\sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$ using asymptotic analysis *)
Entropy1[r_] := Entropy1[r] = Log2[r!/3]

(*‘LengthS2’ is $|S'(\varepsilon, G_v^{\star\star})|$*)
LengthS2[v_] := LengthS2[v] = v! + 1

(*‘Entropy2’ is the approximation $\sup_{\omega\in\Omega_{\varepsilon,v}} \sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega), \psi)))$ using asmyptotic analysis *)
Entropy2[v_] := Entropy2[v] = Log2[v!]

q = 35; (*We want q as large as possible; however, this is limited by
computation time.*)

(*Below is the process of solving ‘TableLowAlphr’ which is $\underline{|S'(\varepsilon, G_r^\star)|}$*)
LowAlphValuesr = Table[
  {sol1[r_] :=
     sol1[r] = Reduce[v > 0 && Entropy2[v] <= Entropy1[r], v, Integers],
    LowSampler = Max[v /. Solve[sol1[r], {v}, Integers]],
    LowAlphr = N[LengthS2[LowSampler]]}, {r, 3, q}];
TableLowAlphr = Table[LowAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}]

(*Below is the process of solving ‘TableUpAlphr’ which is $\overline{|S'(\varepsilon, G_r^\star)|}$*)
UpAlphValuesr = Table[
   {sol11[r_] :=
      sol11[r] =
       Reduce[v < 5000 && Entropy2[v] >= Entropy1[r], v, Integers],
     UpSampler = Min[v /. Solve[sol11[r], {v}, Integers]],
     UpAlphr = N[LengthS2[UpSampler]]}, {r, 3, q}];
TableUpAlphr = Table[UpAlphValuesr[[r - 3 + 1, 3]], {r, 3, q}]

(*Below is the process of solving ‘TableLowAlphv’ which is $\underline{|S'(\varepsilon, G_v^{\star\star})|}$*)
LowAlphValuesv = Table[
   {sol2[v_] :=
      sol2[v] =
       Reduce[r > 0 && Entropy1[r] <= Entropy2[v], r, Integers],
```

```
    LowSamplev = Max[r /. Solve[sol2[v], {r}, Integers]],
    LowAlphv = N[LengthS1[LowSamplev]]}, {v, 3, q}];
TableLowAlphv = Table[LowAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}]

(*Below is the process of solving ‘TableUpAlphv’ which is $\overline{|S'(\varepsilon, G_v^{\star\star})|}$*)
UpAlphValuesv = Table[
   {sol21[v_] :=
     sol21[v] =
      Reduce[r < 5000 && Entropy1[r] >= Entropy2[v], r, Integers],
    UpSamplev = Min[r /. Solve[sol21[v], {r}, Integers]],
    UpAlphv = N[LengthS1[UpSamplev]]}, {v, 3, q}];
TableUpAlphv = Table[UpAlphValuesv[[v - 3 + 1, 3]], {v, 3, q}]

a[r_] := a[r] = TableUpAlphr[[r - 3 + 1]] (*This is $\underline{|S'(\varepsilon, G_r^{\star})|}$*)
b[r_] := b[r] = LengthS1[r] (*This is $|S'(\varepsilon, G_r^{\star})|$*)
c[r_] := c[r] = TableLowAlphr[[r - 3 + 1]] (*This is $\overline{|S'(\varepsilon, G_r^{\star})|}$*)

(*‘K1’ is $K(\varepsilon, G_r^{\star})$*)
K1[r_] :=
 K1[r] = N[
   RealAbs[(b[
          r] (1 + Ceiling[(b[
                r] (a[r] + 2 b[r]))/((a[r] + b[r]) (a[r] + b[r] +
                 c[r]))]) (1 + Round[a[r]/b[r]]))/((1 +
           Round[b[r]/c[r]]) (1 + Round[a[r]/c[r]])) - b[r]] + b[r]]

a1[v_] :=
 a1[v] = TableUpAlphv[[v - 3 + 1]] (*This is $\underline{|S'(\varepsilon, G_v^{\star\star})|}$*)
b1[v_] := b1[v] = LengthS2[v] (*This is $|S'(\varepsilon, G_v^{\star\star})|$*)
c1[v_] := c1[v] = TableLowAlphv[[v - 3 + 1]] (*This is $\overline{|S'(\varepsilon, G_v^{\star\star})|}$*)

(*‘K2’ is $K(\varepsilon, G_v^{\star\star})$*)
K2[v_] :=
 K2[v] = N[
   RealAbs[(b1[
          v] (1 + Ceiling[(b1[
                v] (a1[v] + 2 b1[v]))/((a1[v] + b1[v]) (a1[v] +
                 b1[v] + c1[v]))]) (1 + Round[a1[v]/b1[v]]))/((1 +
           Round[b1[v]/c1[v]]) (1 + Round[a1[v]/c1[v]])) - b1[v]] +
    b1[v]]

(*‘Mr’ is $M'(\varepsilon, G_r^{\star})$*)
Mr = Table[N[LengthS1[r] (K1[r] - LengthS1[r])], {r, 3, q - 1}]

(*‘Mv’ is $M'(\varepsilon, G_v^{\star\star})$*)
Mv = Table[N[LengthS2[v] (K2[v] - LengthS2[v])], {v, 3, q - 1}]

(*‘DownS’ is $\underline{S}(\varepsilon, r, v^*, G_v^{\star\star})$*)
DownS = Table[
  LengthS2[Flatten[
      Position[Mv, Max[Select[Mv, # <= Mr[[r - 4 + 2]] &]]]][[1]] +
    4 - 2], {r, 4, q - 3}]

(*‘UpS’ is $\overline{S}(\varepsilon, r, v^*, G_v^{\star\star})$*)
UpS = Table[
  LengthS2[Flatten[
      Position[Mv, Min[Select[Mv, # >= Mr[[r - 4 + 2]] &]]]][[1]] +
    4 - 2], {r, 4, q - 3}]

E1 = 0 (*Constant rate of expansion*)
dimH = 0 (*Hausdorff Dimension of A*)
Ar = 1(*The smallest 1-dimensional box that covers $A_r^{\star\star\star}$ is [0,1] which
has a length/‘area’ of one *)

(*‘V’ is $V(\varepsilon, G_r^{\star})$ or eq. 135. Note, n is the dimension
of n-Euclidean Plane for which A is a subset*)
V[r_] := V[
   r] = V[r_, n_] :=
 V[r, n] =
```

```
  Ceiling[(Ar^(1 - Sign[E1])) (E1 + (1 - Sign[E1])) ((Pi^(n/2))/
        Gamma[n/2 + 1]) (r!^(n -
          dimH)) (r^(Sign[E1] (dimH - Sign[dimH] + 1))) + (1 -
        Sign[dimH])
      Simplify[V[r]]/eps]/LengthS1[r]

(* We couldn't add v, v* or convert this to a limit due to
limitations of the programming *)
ChoiceFunction =
 Table[N[((Sign[Mr[[r - 5 + 2]]] UpS[[r - 5 + 2]])/(LengthS1[
         r]) - (LengthS1[r]/LengthS2[r])^(-V[r,1]))*((Sign[Mr[[r - 5 + 2]]] DownS[[
          r - 5 + 2]])/(LengthS1[r]) - (LengthS1[r]/
         LengthS2[r])^(-V[r,1]))], {r, 5, q - 3}]
```

7. QUESTIONS

1. Does §6 answer the **leading question** in §3.1
2. Using thm. 9, when f is defined in §2.1, does $\mathbb{E}[f_r^\star]$ have a finite value?
3. Using thm. 9, when f is defined in §2.2, does $\mathbb{E}[f_r^\star]$ have a finite value?
4. If there's no time to check questions 1, 2 and 3, see §4.

8. APPENDIX OF §5.3.1

8.1. **Example of §5.3.1, step 1.**

Suppose

1. $A = \mathbb{R}$
2. When defining $f : A \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \le x < 0.5 \\ 0.5 & 0.5 \le x \end{cases} \tag{148}$$

3. $(G_r^\star)_{r\in\mathbb{N}} = (\{(x, f(x)) : -r \le x \le r\})_{r\in\mathbb{N}}$

Then one example of $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$, using §5.3.1 step 1, (where $G_1^\star = (\{(x, f(x)) : -1 \le x \le 1\})_{r\in\mathbb{N}}$) is:

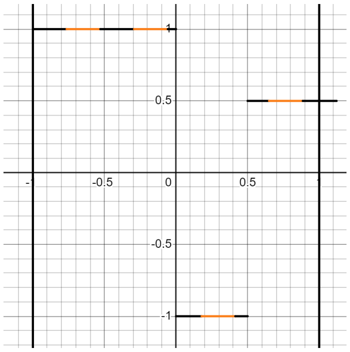
$$\begin{gathered}\left\{\left\{(x,f(x)) : -1 \le x < \frac{\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{\sqrt{2}-6}{6} \le x < \frac{2\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{2\sqrt{2}-6}{6} \le x < \frac{3\sqrt{2}-6}{6}\right\}\right. \\ \left\{(x,f(x)) : \frac{3\sqrt{2}-6}{6} \le x < \frac{4\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{4\sqrt{2}-6}{6} \le x < \frac{5\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{5\sqrt{2}-6}{6} \le x < \frac{6\sqrt{2}-6}{6}\right\} \\ \left.\left\{(x,f(x)) : \frac{6\sqrt{2}-6}{6} \le x < \frac{7\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{7\sqrt{2}-6}{6} \le x < \frac{8\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{8\sqrt{2}-6}{6} \le x \le \frac{9\sqrt{2}-6}{6}\right\}\right\}\end{gathered} \tag{149}$$

Note, the length of each partition is $\sqrt{2}/6$, where the borders could be approximated as:

$$\begin{gathered}\{\,\{(x,f(x)) : -1 \le x < -.764\}, \{(x,f(x)) : -.764 \le x < -.528\}, \{(x,f(x)) : -.528 \le x < -.293\} \\ \{(x,f(x)) : -.293 \le x < -.057\}, \{(x,f(x)) : -.057 \le x < .178\}, \{(x,f(x)) : .178 \le x < .414\} \\ \{(x,f(x)) : .414 \le x < .65\}, \{(x,f(x)) : .65 \le x < .886\}, \{(x,f(x)) : .886 \le x \le 1.121\}\,\}\end{gathered} \tag{150}$$

which is illustrated using *alternating* orange/black lines of equal length covering $G_1^\star$ (i.e., the black vertical lines are the smallest and largest x-cooridinates of $G_1^\star$).

FIGURE 3. The alternating orange *&* black lines are the "covers" and the vertical lines are the boundaries of $G_1^\star$.



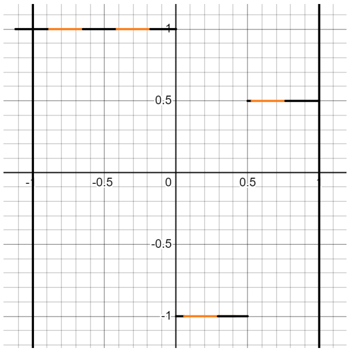
(Note, the alternating covers in fig. 3 satisfy step (1) of §5.3.1, because the Hausdorff measure *in its dimension* of the covers is $\sqrt{2}/6$ and there are 9 covers over-covering $G_1^\star$: i.e.,

Definition 1 (**Minimum Covers of Measure** $\varepsilon = \sqrt{2}/6$ **covering** $G_1^\star$)**.** *We can compute the minimum covers of* $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$*, using the formula:*

$$\lceil \mathcal{H}^{dim_H(G_1^\star)}(G_1^\star)/(\sqrt{2}/6)\rceil$$

where $\lceil \mathcal{H}^{\mathrm{dim_H}(G_1^\star)}(G_1^\star)/(\sqrt{2}/6)\rceil = \lceil \mathrm{Length}([-1,1])/(\sqrt{2}/6)\rceil = \lceil 2/(\sqrt{2}/6)\rceil = \lceil 6\sqrt{2}\rceil = \lceil 6(1.4)\rceil = \lceil 8+.4\rceil = 9)$. Note there are other examples of $\mathbf{C}(\sqrt{2}/6, G_1^\star, \omega)$ for different ω. Here is another case:

FIGURE 4. This is similar to figure 3, except the start-points of the covers are shifted all the way to the left.



which can be defined (see eq. 149 for comparison):

$$\begin{aligned}\Bigg\{&\left\{(x,f(x)):\frac{6-9\sqrt{2}}{6}\leq x<\frac{6-8\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-8\sqrt{2}}{6}\leq x<\frac{6-7\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-7\sqrt{2}}{6}\leq x<\frac{6-6\sqrt{2}}{6}\right\}\\ &\left\{(x,f(x)):\frac{6-6\sqrt{2}}{6}\leq x<\frac{6-5\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-5\sqrt{2}}{6}\leq x<\frac{6-4\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-4\sqrt{2}}{6}\leq x<\frac{6-3\sqrt{2}}{6}\right\}\\ &\left\{(x,f(x)):\frac{6-3\sqrt{2}}{6}\leq x<\frac{6-2\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-2\sqrt{2}}{6}\leq x<\frac{6-\sqrt{2}}{6}\right\},\left\{(x,f(x)):\frac{6-\sqrt{2}}{6}\leq x\leq 1\right\}\Bigg\}\end{aligned} \tag{151}$$

In the case of $G_1^\star$, there are uncountable *different covers* $\mathbf{C}(\sqrt{2}/6, G_1^\star, \omega)$ which can be used. For instance, when $0 \le \alpha \le (12 - 9\sqrt{2})/6$ (i.e., $\omega = \alpha + 1$) consider:

$$\begin{gathered}\left\{\left\{(x, f(x)) : \alpha - 1 + \alpha \le x < \alpha + \frac{\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \alpha + \frac{\sqrt{2}-6}{6} \le x < \alpha + \frac{2\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \alpha + \frac{2\sqrt{2}-6}{6} \le x < \alpha + \frac{3\sqrt{2}-6}{6}\right\}\right. \\ \left\{(x, f(x)) : \alpha + \frac{3\sqrt{2}-6}{6} \le x < \alpha + \frac{4\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \alpha + \frac{4\sqrt{2}-6}{6} \le x < \alpha + \frac{5\sqrt{2}-6}{6}\right\}, \\ \left\{(x, f(x)) : \alpha + \frac{5\sqrt{2}-6}{6} \le x < \alpha + \frac{6\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \alpha + \frac{6\sqrt{2}-6}{6} \le x < \alpha + \frac{7\sqrt{2}-6}{6}\right\} \\ \left.\left\{(x, f(x)) : \alpha + \frac{7\sqrt{2}-6}{6} \le x < \alpha + \frac{8\sqrt{2}-6}{6}\right\}, \left\{(x, f(x)) : \alpha + \frac{8\sqrt{2}-6}{6} \le x \le \alpha + \frac{9\sqrt{2}-6}{6}\right\}\right\}\end{gathered} \tag{152}$$

When $\alpha = 0$ and $\omega = 1$, we get figure 3 and when $\alpha = (12 - 9\sqrt{2})/6$ and $\omega = (18 - 9\sqrt{2})/6$, we get figure 4

8.2. **Example of §5.3.1, step 2.** . Suppose:

(1) $A = \mathbb{R}$
(2) When defining $f : A \to \mathbb{R}$: i.e.,

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \le x < 0.5 \\ 0.5 & 0.5 \le x \end{cases} \tag{153}$$

(3) $(G_r^\star)_{r \in \mathbb{N}} = (\{(x, f(x)) : -r \le x \le r\})_{r \in \mathbb{N}}$
(4) $G_1^\star = \{(x, f(x)) : -1 \le x \le 1\}$
(5) $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$, using eq. 150 and fig. 3, which is *approximately*

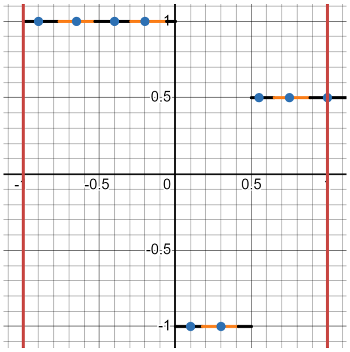
$$\begin{gathered}\{\, \{(x, f(x)) : -1 \le x < -.764\}, \{(x, f(x)) : -.764 \le x < -.528\}, \{(x, f(x)) : -.528 \le x < -.293\} \\ \{(x, f(x)) : -.293 \le x < -.057\}, \{(x, f(x)) : -.057 \le x < .178\}, \{(x, f(x)) : .178 \le x < .414\} \\ \{(x, f(x)) : .414 \le x < .65\}, \{(x, f(x)) : .65 \le x < .886\}, \{(x, f(x)) : .886 \le x \le 1.121\} \,\}\end{gathered} \tag{154}$$

Then, an example of $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{155}$$

Below, we illustrate the sample: i.e., the set of all blue points *in each orange and black line of* $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$ covering $G_1^\star$:

FIGURE 5. The blue points are the "sample points", the alternative black and orange lines are the "covers", and the red lines are the *smallest* & *largest* x-coordinates of $G_1^\star$.



Note, there are multiple samples that can be taken, as long as one sample point is taken from each cover in $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$.

8.3. **Example of §5.3.1, step 3.** Suppose

(1) $A = \mathbb{R}$

(2) When defining $f : A \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \leq x < 0.5 \\ 0.5 & 0.5 \leq x \end{cases} \tag{156}$$

(3) $(G_r^{\star})_{r \in \mathbb{N}} = (\{(x, f(x)) : -r \leq x \leq r\})_{r \in \mathbb{N}}$

(4) $G_1^{\star} = \{(x, f(x)) : -1 \leq x \leq 1\}$

(5) $\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1)$, using eq. 150 and fig. 3, is approx.

$$\begin{aligned} \{\, &\{(x, f(x)) : -1 \leq x < -.764\}, \{(x, f(x)) : -.764 \leq x < -.528\}, \{(x, f(x)) : -.528 \leq x < -.293\} \\ &\{(x, f(x)) : -.293 \leq x < -.057\}, \{(x, f(x)) : -.057 \leq x < .178\}, \{(x, f(x)) : .178 \leq x < .414\} \\ &\{(x, f(x)) : .414 \leq x < .65\}, \{(x, f(x)) : .65 \leq x < .886\}, \{(x, f(x)) : .886 \leq x \leq 1.121\} \,\} \end{aligned} \tag{157}$$

(6) $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)$, using eq. 155, is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{158}$$

Therefore, consider the following process:

8.3.1. *Step 3a.* If $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)$ is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{159}$$

suppose $x_0 = (-.9, 1)$. Note, the following:

1. $x_1 = (-.65, 1)$ is the next point in the "pathway" since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)$ with the smallest 2-d Euclidean distance to x_0 instead of x_0.
2. $x_2 = (-.4, 1)$ is the third point since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)$ with the smallest 2-d Euclidean distance to x_1 instead of x_0 and x_1.
3. $x_3 = (-.2, 1)$ is the fourth point since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)$ with the smallest 2-d Euclidean distance to x_2 instead of x_0, x_1, and x_2.
4. we continue this process, where the "**pathway**" of $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)$ is:

$$(-.9, 1) \to (-.65, 1) \to (-.4, 1) \to (-.2, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1) \tag{160}$$

Note 10. *If more than one point has the minimum* 2*-d Euclidean distance from* x_0*,* x_1*,* x_2*, etc. take all potential pathways: e.g., using the sample in eq. 159, if* $x_0 = (-.65, 1)$*, then since* $(-.9, 1)$ *and* $(-.4, 1)$ *have the smallest Euclidean distance to* $(-.65, 1)$*, take* ***two*** *pathways:*

$$(-.65, 1) \to (-.9, 1) \to (-.4, 1) \to (-.2, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1)$$

and also:

$$(-.65, 1) \to (-.4, 1) \to (-.2, 1) \to (-.9, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1)$$

8.3.2. *Step 3b.* Next, take the length of all line segments in each pathway. In other words, suppose $d(P, Q)$ is the n-th dim. Euclidean distance between points $P, Q \in \mathbb{R}^n$. Using the pathway in eq. 160, we want:

$$\begin{aligned} &\{d((-.9, 1), (-.65, 1)), d((-.65, 1), (-.4, 1)), d((-.4, 1), (-.2, 1)), d((-.2, 1), (.55, .5)), \\ &d((.55, .5), (.75, .5)), d((.75, .5), (1, .5)), d((1, .5), (.3, -1)), d((.3, -1), (.1, -1))\} \end{aligned} \tag{161}$$

Whose distances can be approximated as:

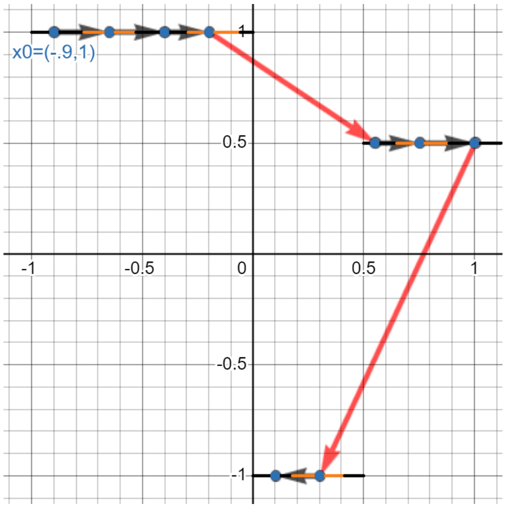
$$\{.25, .25, .2, .901389, .2, .25, 1.655295, .2\}$$

Also, we see the outliers [5] are .901389 and 1.655295 (i.e., notice that the outliers are more prominent for $\varepsilon \ll \sqrt{2}/6$). Therefore, remove .901389 and 1.655295 from our set of lengths:

$$\{.25, .25, .2, .2, .25, .2\}$$

This is illustrated using:

FIGURE 6. The black arrows are the "pathways" whose lengths aren't outliers. The length of the red arrows in the pathway are outliers.



Hence, when $x_0 = (-.9, 1)$, using §5.3.1 step 3b & eq. 159, we note:

$$\mathcal{L}((-.9, 1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)) = \{.25, .25, .2, .2, .25, .2\} \tag{162}$$

8.3.3. *Step 3c.* To convert the set of distances in eq. 162 into a probability distribution, we take:

$$\sum_{x \in \{.25,.25,.2,.2,.25,.2\}} x = .25 + .25 + .2 + .2 + .25 + .2 = 1.35 \tag{163}$$

Then divide each element in $\{.25, .25, .2, .2, .25, .2\}$ by 1.35

$$\{.25/(1.35), .25/(1.35), .2/(1.35), .2/(1.35), .25/(1.35), .2/(1.35)\}$$

which gives us the probability distribution:

$$\{5/27, 5/27, 4/27, 4/27, 5/27, 4/27\}$$

Hence,

$$\mathbb{P}(\mathcal{L}((-.9, 1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1))) = \{5/27, 5/27, 4/27, 4/27, 5/27, 4/27\} \tag{164}$$

8.3.4. *Step 3d.* Take the shannon entropy of eq. 164:

$$\begin{aligned}
&\mathrm{E}(\mathbb{P}(\mathcal{L}((-.9, 1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)))) = \\
&\sum_{x \in \mathbb{P}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1)))} -x \log_2 x = \sum_{x \in \{5/27,5/27,4/27,4/27,5/27,4/27\}} -x \log_2 x = \\
&-(5/27)\log_2(5/27) - (5/27)\log_2(5/27) - (4/27)\log_2(4/27) - (4/27)\log_2(4/27) - (5/27)\log_2(5/27) - (4/27)\log_2(5/27) = \\
&-(15/27)\log_2(5/27) - (12/27)\log_2(4/27) \approx 2.57604
\end{aligned}$$

We shorten $\mathrm{E}(\mathbb{P}(\mathcal{L}((-.9, 1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1))))$ to $\mathrm{E}(\mathcal{L}((-.9, 1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)))$, giving us:

$$\mathrm{E}(\mathcal{L}((-.9, 1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1))) \approx 2.57604 \tag{165}$$

8.3.5. *Step 3e.* Take the entropy, w.r.t all pathways, of the sample:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{166}$$

In other words, we'll compute:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1))) = \sup_{x_0 \in \mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1)} \mathrm{E}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^{\star}, 1), 1)))$$

We do this by repeating §8.3.1-§8.3.4 for different $x_0 \in \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)))$ (i.e., in the equation with multiple values, see note 10)

$$\mathrm{E}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 2.57604 \tag{167}$$

$$\mathrm{E}(\mathcal{L}((-.65,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 2.3131, 2.377604 \tag{168}$$

$$\mathrm{E}(\mathcal{L}((-.4,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 2.3131 \tag{169}$$

$$\mathrm{E}(\mathcal{L}((-.2,-1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 2.57604 \tag{170}$$

$$\mathrm{E}(\mathcal{L}((-.1,-1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 1.86094 \tag{171}$$

$$\mathrm{E}(\mathcal{L}((-.3,-1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 1.85289 \tag{172}$$

$$\mathrm{E}(\mathcal{L}((.55,.5), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 2.08327 \tag{173}$$

$$\mathrm{E}(\mathcal{L}((.75,.5), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 2.31185 \tag{174}$$

$$\mathrm{E}(\mathcal{L}((1,.5), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 2.2622 \tag{175}$$

Hence, since the largest value out of eq. 167-175 is 2.57604:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) = \sup_{x_0 \in \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)} \mathrm{E}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 2.57604$$

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