DEFINING AN UNIQUE, SATISFYING EXPECTED VALUE FROM CHOSEN SEQUENCES OF BOUNDED FUNCTIONS CONVERGING TO A EVERYWHERE SURJECTIVE FUNCTION

BHARATH KRISHNAN

Abstract. Let $n \in \mathbb{N}$ and suppose function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel. In this paper, we average a everywhere surjective f whose graph has zero Hausdorff measure in its dimension, taking finite values only. Since, by definition, the expected value of f is indeterminate, choose any sequence of bounded functions converging to f with the same satisfying and finite expected value. Note, "satisfying" is determined by a leading question in §4.2 which uses a rate of expansion of the sequence of each bounded functions' graph (§5.1) and a "measure" involving covers, samples, pathways, and entropy (§5.2).

Let $n \in \mathbb{N}$ and suppose function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel. Let $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension, where $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure *in its dimension* on the Borel σ-algebra.

1. Motivation

Suppose, we define everywhere surjective f:

> Let (A, T) be a standard topology. A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is *everywhere surjective* from A to $\mathbb{R}$, if $f[V] = \mathbb{R}$ for every $V \in \mathrm{T}$.

If f is everywhere surjective, whose graph has zero Hausdorff measure in its dimension (e.g., [2]), we want a unique, satisfying (§4) average of f, taking finite values only. However, the expected value of f:

$$\mathbb{E}[f] = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A)}(A)} \int_A f \, d\mathcal{H}^{\dim_{\mathrm{H}}(A)}$$

is undefined since the integral of f is undefined: i.e., the graph of f has Hausdorff dimension $n+1$ with zero $(n+1)$-dimensional Hausdorff measure. Thus, w.r.t a reference point $C \in \mathbb{R}^{n+1}$ (§4.1), choose any sequence of bounded functions converging to f (§2.1) with the same satisfying (§4) and finite expected value (§2.2).

To account for the previous sentence, we solve two additional problems involving prevalence and shyness (§2.3, §3.1):

1. If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where $\mathbb{E}[f]$ is finite, then F is shy (§2.3).
2. If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where two sequences of bounded functions that converge to f (§2.1) have different expected values (§2.2), then F is prevelant (§2.3).

Note, with this paper [3], we gain more insight into the motivation.

2. Preliminaries

2.1. Definition of sequences of Functions Converging to f.

Let $n \in \mathbb{N}$ and suppose function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel.

The sequence of functions $(f_r)_{r\in\mathbb{N}}$, where $(A_r)_{r\in\mathbb{N}}$ is a sequence of sets and function $f_r : A_r \to \mathbb{R}$, converges to f when:

> For any $x \in A$, there exists a sequence $\mathbf{x} \in A_r$ *s.t.* $\mathbf{x} \to (x_1, \cdots, x_n)$ and $f_r(\mathbf{x}) \to f(x_1, \cdots, x_n)$.

This is equivalent to:

$$(f_r, A_r) \to (f, A)$$

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2.2. **Expected Value of Sequences of Functions Converging to f.** Thus, suppose:

- $(f_r, A_r) \to (f, A)$ (§2.1)
- $|\cdot|$ is the absolute value
- $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension
- $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra
- the integral is defined, w.r.t the Hausdorff measure in its dimension

The expected value of $(f_r)_{r\in\mathbb{N}}$ is a real number $\mathbb{E}[f_r]$, when the following is true:

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N}) \left(r \geq N \Rightarrow \left| \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}(A_r)} \int_{A_r} f_r \, d\mathcal{H}^{\dim_{\mathrm{H}}(A_r)} - \mathbb{E}[f_r] \right| < \epsilon \right) \tag{1}$$

when no such $\mathbb{E}[f_r]$ exists, $\mathbb{E}[f_r]$ is infinite or undefined. (If the graph of f has zero Hausdorff measure in its dimension, replace $\mathcal{H}^{\dim_{\mathrm{H}}(A_r)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33].)

2.3. **Definition of Prevalent and Shy Sets.** Here is the source [4, def. 3.1]:

A Borel set $E \subset X$ is said to be **prevalent** if there exists a Borel measure μ on X such that:

(1) $0 < \mu(C) < \infty$ for some compact subset C of X, and
(2) the set $E + x$ has full μ-measure (that is, the complement of $E + x$ has measure zero) for all $x \in X$.

More generally, a subset F of X is prevalent if F contains a prevalent Borel Set.

Additionally:

- The complement of a prevelant set is a **shy** set.

Thus, notice:

- If $F \subset X$ is *prevelant*, we say "**almost every**" element of X lies in F.
- If $F \subset X$ is *shy*, we say "**almost no**" element of X lies in F.

3. Motivation to Answer §1

Let $n \in \mathbb{N}$ and suppose function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, where A and f are Borel. Let $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension, where $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure *in its dimension* on the Borel σ-algebra.

3.1. **Problems.** If $\mathbb{E}[f]$ is the expected value of f, w.r.t the Hausdorff measure in its dimension,

$$\mathbb{E}[f] = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(A)}(A)} \int_A f \, d\mathcal{H}^{\dim_{\mathrm{H}}(A)}$$

then suppose:

- $\mathbf{B}(X)$ is the set of all bounded Borel subsets of set X
- $\mathfrak{B}(X)$ is the set of all bounded Borel functions with a domain X

therefore:

(1) If f is everywhere surjective (§1), whose graph has zero Hausdorff measure in its dimension (e.g., [2]), $\mathbb{E}[f]$ is undefined and non-finite.
(2) If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where $\mathbb{E}[f]$ is finite, then F is shy (§2.3).
(3) For all $r, v \in \mathbb{N}$, suppose $A_r, B_v \in \mathbf{B}(\mathbb{R}^n)$, where $f_r \in \mathfrak{B}(A_r)$ and $g_v \in \mathfrak{B}(B_v)$. If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where $(f_r, A_r), (g_v, B_v) \to (f, A)$ (§2.1) and $\mathbb{E}[f_r] \neq \mathbb{E}[g_v]$ (§2.2), then F is prevalent (§2.3).

3.2. **Approach to Solving the Statements In §3.1.** To solve the statements in §3.1 *at once*, consider the following:

> For all $r \in \mathbb{N}$, suppose $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$ is an arbitrary set, where $A_r \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r)$ is an arbitrary set, such that $f_r \in \mathscr{B}$. If $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where $(f_r, A_r) \to (f, A)$ (§2.1) and $\mathbb{E}[f_r]$ (§2.2) is unique, satisfying (§4), and finite, then F should be:
> (1) a prevalent (§2.3) subset of $\mathbb{R}^A$
> (2) If not prevalent (§2.3) then neither a prevalent (§2.3) nor shy (§2.3) subset of $\mathbb{R}^A$.

Question: How do we define "satisfying" in §3.2, w.r.t a reference point $C \in \mathbb{R}^{n+1}$, so that $\mathbb{E}[f_r]$ satisfies §1? (In §4, we give a partial solution to this question, where $(f_r)_{r\in\mathbb{N}} = (f^{\star}_r)_{r\in\mathbb{N}}$.)

4. Example of Partial Solution Explaining The Term "Satisfying" In §1 and §3.2

We ask a leading question in §4.2 with an answer that should solve the question in §3.2.

4.1. Preliminaries. Suppose, for all $r, v \in \mathbb{N}$, there exists arbitrary set $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, where $A_r^\star \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r^\star)$ (§3.1) such that:

- $f_r^\star \in \mathscr{B}$
- $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$ and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star}) \cup (\mathfrak{B}(A_r^\star) \setminus \mathscr{B})$
- $(G_r^\star)_{r\in\mathbb{N}} = (\text{graph}(f_r^\star))_{r\in\mathbb{N}}$ is the sequence of the graph of each $f_r^\star$
- $\square$ is the logical symbol for "it's necessary"
- C is a reference point in $\mathbb{R}^{n+1}$ (e.g., the origin)
- E is the fixed, expected rate of expansion of $(G_r^\star)_{r\in\mathbb{N}}$ w.r.t a reference point C: e.g., $E = 1$
- $\mathcal{E}(C, G_r^\star)$ is the actual rate of expansion of $(G_r^\star)_{r\in\mathbb{N}}$ w.r.t a reference point C (§5.1)

4.2. Leading Question.

Does there exist a unique choice function, which for all $r \in \mathbb{N}$, chooses a unique set $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, where $A_r^\star \in \mathcal{B}$ and a unique set $\mathscr{B} \subset \mathfrak{B}(A_r^\star)$ (§3.1) such that $f_r^\star \in \mathscr{B}$, where:

(1) $(f_r^\star, A_r^\star) \to (f, A)$ (§2.1)

(2) For all $v \in \mathbb{N}$, where for all $A_v^{\star\star} \in \mathbf{B}(\mathbb{R}^n) \setminus \mathcal{B}$ and $f_v^{\star\star} \in \mathfrak{B}(A_v^{\star\star}) \cup (\mathfrak{B}(A_r^\star) \setminus \mathscr{B})$, assuming $(f_v^{\star\star}, A_v^{\star\star}) \to (f, A)$ (§2.1), the "measure" (§5.2.1, §5.2.2) of $(G_r^\star)_{r\in\mathbb{N}} = (\text{graph}(f_r^\star))_{r\in\mathbb{N}}$ (§4.1) must increase at a rate linear or superlinear to that of $(G_v^{\star\star})_{v\in\mathbb{N}} = (\text{graph}(f_v^{\star\star}))_{v\in\mathbb{N}}$ (§4.1)

(3) $\mathbb{E}[f_r^\star]$ is unique and finite (§2.2)

(4) For some $A_r^\star \in \mathcal{B}$ and $f_r^\star \in \mathscr{B}$ satisfying (1), (2) and (3), when f is unbounded (i.e, skip (4) when f is bounded), for all $s \in \mathbb{N}$ and for any set $\mathcal{B}' \subset \mathbf{B}(\mathbb{R}^n)$, where $A_s^{\star\star\star} \in \mathcal{B}'$, and for any set $\mathscr{B}' \subset \mathfrak{B}(A_s^{\star\star\star})$, where $\star \mapsto \star\star\star$, $r \mapsto s$, $\mathcal{B} \mapsto \mathcal{B}'$, and $\mathscr{B} \mapsto \mathscr{B}'$ in (1), (2) and (3), s.t. $\neg\square(\mathbb{E}[f_r^\star] = \mathbb{E}[f_s^{\star\star\star}])$ (§2.2, §4.1), when $f_s^{\star\star\star} \in \mathscr{B}'$ satisfies (1), (2) and (3):
   - If the absolute value is $|\cdot|$ and the $(n+1)$-th coordinate of C (§4.1) is x_{n+1}, $|\mathbb{E}[f_r^\star] - x_{n+1}| \le |\mathbb{E}[f_s^{\star\star\star}] - x_{n+1}|$ (§2.2)
   - If $r \in \mathbb{N}$, then for all linear $s_1 : \mathbb{N} \to \mathbb{N}$, where $s = s_1(r)$ and the Big-O notation is $\mathcal{O}$, there exists a function $K : \mathbb{R} \to \mathbb{R}$, where the absolute value is $|\cdot|$ and (§4.1):

$$\begin{aligned}|\mathcal{E}(C, G_r^\star) - E| =& \mathcal{O}(K(|\mathcal{E}(C, G_s^{\star\star\star}) - E|)) \\ =& \mathcal{O}(K(|\mathcal{E}(C, G_{s_1(r)}^{\star\star\star}) - E|))\end{aligned}$$

   such that:

$$0 \le \lim_{x\to+\infty} K(x)/x < +\infty$$

   In simpler terms, "the rate of divergence" of $|\mathcal{E}(C, G_r^\star) - E|$ (§4.1) is *less than or equal* to "the rate of divergence" of $|\mathcal{E}(C, G_s^{\star\star\star}) - E|$ (§4.1).

(5) When set $F \subset \mathbb{R}^A$ is the set of all $f \in \mathbb{R}^A$, where a choice function chooses a collection $\mathcal{B} \subset \mathbf{B}(\mathbb{R}^n)$, where $A_r^\star \in \mathcal{B}$ and $\mathscr{B} \subset \mathfrak{B}(A_r^\star)$ such that $f_r^\star \in \mathscr{B}$ satisfies (1), (2), (3) and (4), then F should be:
   (a) a prevalent (§2.3) subset of $\mathbb{R}^A$
   (b) If not (a), then neither a prevalent (§2.3) nor shy (§2.3) subset of $\mathbb{R}^A$

(6) Out of all choice functions that satisfy (1), (2), (3), (4) and (5), we choose the one with the simplest form, meaning for each choice function fully expanded, we take the one with the fewest variables/numbers?

Question 2 How do we improve §4.2, so the answer satisfies §3.2 and $\mathbb{E}[f_r^\star]$ satisfies §1?

5. Appendix

5.1. Actual Rate of Expansion of $(G_r^\star)$ w.r.t a reference point C. Suppose:

(1) $(G_r^\star)_{r\in\mathbb{N}}$ is a sequence of the graph of each $f_r^\star$ (§4.1)
(2) C is a reference point in $\mathbb{R}^{n+1}$
(3) $Q, R \in \mathbb{R}^{n+1}$

(4) $Q = (q_1, \cdots, q_{n+1})$ and $R = (r_1, \cdots, r_{n+1})$, where:

$$Q - R = (q_1 - r_1, \cdots, q_{n+1} - r_{n+1})$$

(5) $||Q|| = \sqrt{q_1^2 + \cdots + q_{n+1}^2}$ and $||R|| = \sqrt{r_1^2 + \cdots + r_{n+1}^2}$
(6) $C - G_r^\star = \{C - y : y \in G_r^\star\}$
(7) $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension
(8) $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure in its dimension on the Borel σ-algebra

For any $r \in \mathbb{N}$, take the $(n+1)$-dimensional Euclidean distance between a reference point $C \in \mathbb{R}^{n+1}$ and each point in $G_r^\star$:

$$\mathcal{G}(C, G_r^\star) = \{||C - y|| : y \in G_r^\star\}$$

then average $\mathcal{G}(C, G_r^\star)$:

$$\operatorname{Avg}(\mathcal{G}(C, G_r^\star)) = \frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(C - G_r^\star)}(C - G_r^\star)} \int_{C - G_r^\star} ||(x_1, \cdots, x_{n+1})|| \, d\mathcal{H}^{\dim_{\mathrm{H}}(C - G_r^\star)}$$

where **the actual rate of expansion of** $(G_r^\star)_{r\in\mathbb{N}}$ is:

$$\mathcal{E}(C, G_r^\star) = \operatorname{Avg}(\mathcal{G}(C, G_{r+1}^\star)) - \operatorname{Avg}(\mathcal{G}(C, G_r^\star))$$

If $\mathcal{E}(C, G_r^\star)$ is undefined, replace the Hausdorff measure $\mathcal{H}^{\dim_{\mathrm{H}}(C - G_r^\star)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33]

5.2. **Defining the "Measure".**

5.2.1. *Preliminaries.* We define the "**measure**" of $(G_r^\star)_{r\in\mathbb{N}}$, in §5.2.2, which is the sequence of the graph of each $f_r^\star$ (§4.1). To understand this "measure", continue reading.

(1) For every $r \in \mathbb{N}$, "over-cover" $G_r^\star$ with minimal, pairwise disjoint sets of equal $\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}$ measure. (We denote the equal measures ε, where the former sentence is defined $\mathbf{C}(\varepsilon, G_r^\star, \omega)$: i.e., $\omega \in \Omega_{\varepsilon,r}$ enumerates all collections of these sets covering $G_r^\star$. In case this step is unclear, see §6.1. Moreover, when there exists a $r \in \mathbb{N}$, where $\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}(G_r^\star) = 0$, replace the Hausdorff measure $\mathcal{H}^{\dim_{\mathrm{H}}(G_r^\star)}$ with the generalized Hausdorff measure $\mathscr{H}^{\phi^{\mu}_{h,g}(q,t)}$ [1, p.26-33].)
(2) For every ε, r and ω, take a sample point from each set in $\mathbf{C}(\varepsilon, G_r^\star, \omega)$. The set of these points is "the sample" which we define $\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)$: i.e., $\psi \in \Psi_{\varepsilon,r,\omega}$ enumerates all possible samples of $\mathbf{C}(\varepsilon, G_r^\star, \omega)$. (If this is unclear, see §6.2.)
(3) For every ε, r, ω and ψ,
    (a) Take a "pathway" of line segments: we start with a line segment from arbitrary point x_0 of $\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)$ to the sample point with the smallest $(n+1)$-dimensional Euclidean distance to x_0 (i.e., when more than one sample point has the smallest $(n+1)$-dimensional Euclidean distance to x_0, take either of those points). Next, repeat this process until the "pathway" intersects with every sample point once. (In case this is unclear, see §6.3.1.)
    (b) Take the set of the length of all segments in (a), except for lengths that are outliers (i.e., for any constant $C > 0$, the outliers are more than C times the interquartile range of the length of all line segments as $r \to \infty$ or $\varepsilon \to 0$). Define this $\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi))$. (If this is unclear, see §6.3.2.)
    (c) Multiply remaining lengths in the pathway by a constant so they add up to one (i.e., a probability distribution). This will be denoted $\mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$. (In case this is unclear, see §6.3.3)
    (d) Take the shannon entropy of step (c). We define this:

$$\mathrm{E}(\mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))) = \sum_{x \in \mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))} -x \log_2 x$$

    which will be *shortened* to $\mathrm{E}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$. (If this is unclear, see §6.3.4.)
    (e) Maximize the entropy w.r.t all "pathways". This we will denote:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi))) = \sup_{x_0 \in \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)} \mathrm{E}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, G_r^\star, \omega), \psi)))$$

    (In case this is unclear, see §6.3.5.)

(4) Therefore, the **maximum entropy**, using (1) and (2) is:

$$\mathrm{E}_{\max}(\varepsilon, r) = \sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))$$

5.2.2. *What Am I "Measuring"?* We define $(G_r^{\star})_{r\in\mathbb{N}}$ and $(G_v^{\star\star})_{v\in\mathbb{N}}$, which respectively are the sequences of the graph for each of the bounded functions $f_r^{\star}$ and $f_v^{\star\star}$ (§4.1). Hence, for **constant** ε and *cardinality* $|\cdot|$

(a) Using (2) and (3e) of section 5.2.1, suppose:

$\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|} =$
$\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \geq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))\right\}$

then (using $\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|}$) we get:

$$\overline{\alpha}\,(\varepsilon, r, \omega, \psi) = \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|}/\,|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi))|$$

(b) Also, using (2) and (3e) of section 5.2.1, suppose:

$\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|} =$
$\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi')\right| : v \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,v},\ \psi' \in \Psi_{\varepsilon,v,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_v^{\star\star}, \omega'), \psi'))) \leq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)))\right\}$

then (using $\underline{|\mathcal{S}(\mathbf{C}(\epsilon, G_r^{\star}, \omega), \psi)|}$) we also get

$$\underline{\alpha}\,(\varepsilon, r, \omega, \psi) = \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|}/\,|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi))|$$

(1) If using $\overline{\alpha}\,(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}\,(\epsilon, r, \omega, \psi)$ we have:

$$1 < \limsup_{\varepsilon\to 0} \limsup_{r\to\infty} \sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \overline{\alpha}\,(\varepsilon, r, \omega, \psi)\,, \liminf_{\varepsilon\to 0} \liminf_{r\to\infty} \inf_{\omega\in\Omega_{\varepsilon,r}} \inf_{\psi\in\Psi_{\varepsilon,r,\omega}} \underline{\alpha}\,(\varepsilon, r, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(G_r^{\star})_{r\in\mathbb{N}}$ increases at a rate **superlinear** to that of $(G_v^{\star\star})_{v\in\mathbb{N}}$.

(2) If using equations $\overline{\alpha}\,(\varepsilon, v, \omega, \psi)$ and $\underline{\alpha}\,(\varepsilon, v, \omega, \psi)$ (where, using $\overline{\alpha}\,(\varepsilon, r, \omega, \psi)$ and $\underline{\alpha}\,(\varepsilon, r, \omega, \psi)$, we swap r with v and $G_r^{\star}$ with $G_v^{\star\star}$) we get:

$$1 < \limsup_{\varepsilon\to 0} \limsup_{v\to\infty} \sup_{\omega\in\Omega_{\varepsilon,v}} \sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \overline{\alpha}\,(\varepsilon, v, \omega, \psi)\,, \liminf_{\varepsilon\to 0} \liminf_{v\to\infty} \inf_{\omega\in\Omega_{\varepsilon,v}} \inf_{\psi\in\Psi_{\varepsilon,v,\omega}} \underline{\alpha}\,(\varepsilon, v, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(G_r^{\star})_{r\in\mathbb{N}}$ increases at a rate **sublinear** to that of $(G_v^{\star\star})_{v\in\mathbb{N}}$.

(3) If using equations $\overline{\alpha}\,(\varepsilon, r, \omega, \psi)$, $\underline{\alpha}\,(\varepsilon, r, \omega, \psi)$, $\overline{\alpha}\,(\varepsilon, v, \omega, \psi)$, and $\underline{\alpha}\,(\varepsilon, v, \omega, \psi)$, we **both** have:

(a) $\limsup_{\varepsilon\to 0} \limsup_{r\to\infty} \sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,r,\omega}} \overline{\alpha}\,(\varepsilon, r, \omega, \psi)$ or $\liminf_{\varepsilon\to 0} \liminf_{r\to\infty} \inf_{\omega\in\Omega_{\varepsilon,r}} \inf_{\psi\in\Psi_{\varepsilon,r,\omega}} \underline{\alpha}\,(\varepsilon, r, \omega, \psi)$ are equal to zero, one or $+\infty$

(b) $\limsup_{\varepsilon\to 0} \limsup_{v\to\infty} \sup_{\omega\in\Omega_{\varepsilon,v}} \sup_{\psi\in\Psi_{\varepsilon,v,\omega}} \overline{\alpha}\,(\varepsilon, v, \omega, \psi)$ or $\liminf_{\varepsilon\to 0} \liminf_{v\to\infty} \inf_{\omega\in\Omega_{\varepsilon,v}} \inf_{\psi\in\Psi_{\varepsilon,v,\omega}} \underline{\alpha}\,(\varepsilon, v, \omega, \psi)$ are equal to zero, one or $+\infty$

then *what I'm measuring from* $(G_r^{\star})_{r\in\mathbb{N}}$ increases at a rate **linear** to that of $(G_v^{\star\star})_{v\in\mathbb{N}}$.

6. Illustration of §5.2.1

6.1. **Example of §5.2.1, step 1.** Suppose

(1) $A = \mathbb{R}$

(2) When defining $f : A \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \leq x < 0.5 \\ 0.5 & 0.5 \leq x \end{cases} \tag{2}$$

(3) $(G_r^{\star})_{r\in\mathbb{N}} = (\{(x, f(x)) : -r \leq x \leq r\})_{r\in\mathbb{N}}$

Then one example of $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$, using §5.2.1 step 1, (where $G_1^\star = (\{(x, f(x)) : -1 \leq x \leq 1\})_{r \in \mathbb{N}}$) is:

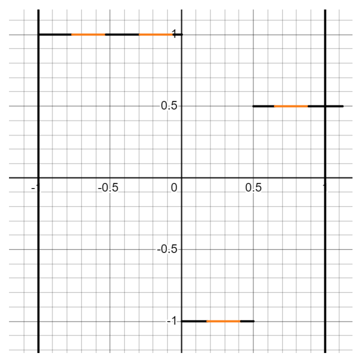
$$\begin{aligned}\Bigg\{ &\left\{(x,f(x)) : -1 \leq x < \frac{\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{\sqrt{2}-6}{6} \leq x < \frac{2\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{2\sqrt{2}-6}{6} \leq x < \frac{3\sqrt{2}-6}{6}\right\} \\ &\left\{(x,f(x)) : \frac{3\sqrt{2}-6}{6} \leq x < \frac{4\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{4\sqrt{2}-6}{6} \leq x < \frac{5\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{5\sqrt{2}-6}{6} \leq x < \frac{6\sqrt{2}-6}{6}\right\} \\ &\left\{(x,f(x)) : \frac{6\sqrt{2}-6}{6} \leq x < \frac{7\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{7\sqrt{2}-6}{6} \leq x < \frac{8\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \frac{8\sqrt{2}-6}{6} \leq x \leq \frac{9\sqrt{2}-6}{6}\right\}\Bigg\}\end{aligned} \tag{3}$$

Note, the length of each partition is $\sqrt{2}/6$, where the borders could be approximated as:

$$\begin{aligned}\{\, &\{(x,f(x)) : -1 \leq x < -.764\}, \{(x,f(x)) : -.764 \leq x < -.528\}, \{(x,f(x)) : -.528 \leq x < -.293\} \\ &\{(x,f(x)) : -.293 \leq x < -.057\}, \{(x,f(x)) : -.057 \leq x < .178\}, \{(x,f(x)) : .178 \leq x < .414\} \\ &\{(x,f(x)) : .414 \leq x < .65\}, \{(x,f(x)) : .65 \leq x < .886\}, \{(x,f(x)) : .886 \leq x \leq 1.121\}\,\}\end{aligned} \tag{4}$$

which is illustrated using *alternating* orange/black lines of equal length covering $G_1^\star$ (i.e., the black vertical lines are the smallest and largest x-cooridinates of $G_1^\star$).

FIGURE 1. The alternating orange & black lines are the "covers" and the vertical lines are the boundaries of $G_1^\star$.



(Note, the alternating covers in fig. 1 satisfy step (1) of §5.2.1, because the Hausdorff measure *in its dimension* of the covers is $\sqrt{2}/6$ and there are 9 covers over-covering $G_1^\star$: i.e.,

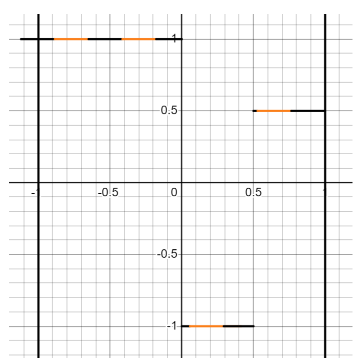
Definition 1 (Minimum Covers of Measure $\varepsilon = \sqrt{2}/6$ covering $G_1^\star$). *We can compute the minimum covers of* $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$*, using the formula:*

$$\lceil \mathcal{H}^{dim_H(G_1^\star)}(G_1^\star)/(\sqrt{2}/6) \rceil$$

where $\lceil \mathcal{H}^{\dim_{\mathrm{H}}(G_1^\star)}(G_1^\star)/(\sqrt{2}/6) \rceil = \lceil \text{Length}([-1,1])/(\sqrt{2}/6) \rceil = \lceil 2/(\sqrt{2}/6) \rceil = \lceil 6\sqrt{2} \rceil = \lceil 6(1.4) \rceil = \lceil 8 + .4 \rceil = 9$).

Note there are other examples of $\mathbf{C}(\sqrt{2}/6, G_1^\star, \omega)$ for different ω. Here is another case:

FIGURE 2. This is similar to figure 1, except the start-points of the covers are shifted all the way to the left.



which can be defined (see eq. 3 for comparison):

$$\begin{aligned}\Bigg\{ &\left\{(x,f(x)) : \frac{6-9\sqrt{2}}{6} \le x < \frac{6-8\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-8\sqrt{2}}{6} \le x < \frac{6-7\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-7\sqrt{2}}{6} \le x < \frac{6-6\sqrt{2}}{6}\right\} \\ &\left\{(x,f(x)) : \frac{6-6\sqrt{2}}{6} \le x < \frac{6-5\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-5\sqrt{2}}{6} \le x < \frac{6-4\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-4\sqrt{2}}{6} \le x < \frac{6-3\sqrt{2}}{6}\right\} \\ &\left\{(x,f(x)) : \frac{6-3\sqrt{2}}{6} \le x < \frac{6-2\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-2\sqrt{2}}{6} \le x < \frac{6-\sqrt{2}}{6}\right\}, \left\{(x,f(x)) : \frac{6-\sqrt{2}}{6} \le x \le 1\right\}\Bigg\}\end{aligned} \tag{5}$$

In the case of $G_1^\star$, there are uncountable *different covers* $\mathbf{C}(\sqrt{2}/6, G_1^\star, \omega)$ which can be used. For instance, when $0 \le \alpha \le (12-9\sqrt{2})/6$ (i.e., $\omega = \alpha + 1$) consider:

$$\begin{aligned}\Bigg\{ &\left\{(x,f(x)) : \alpha - 1 \le x < \alpha + \frac{\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{\sqrt{2}-6}{6} \le x < \alpha + \frac{2\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{2\sqrt{2}-6}{6} \le x < \alpha + \frac{3\sqrt{2}-6}{6}\right\} \\ &\left\{(x,f(x)) : \alpha + \frac{3\sqrt{2}-6}{6} \le x < \alpha + \frac{4\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{4\sqrt{2}-6}{6} \le x < \alpha + \frac{5\sqrt{2}-6}{6}\right\}, \\ &\left\{(x,f(x)) : \alpha + \frac{5\sqrt{2}-6}{6} \le x < \alpha + \frac{6\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{6\sqrt{2}-6}{6} \le x < \alpha + \frac{7\sqrt{2}-6}{6}\right\} \\ &\left\{(x,f(x)) : \alpha + \frac{7\sqrt{2}-6}{6} \le x < \alpha + \frac{8\sqrt{2}-6}{6}\right\}, \left\{(x,f(x)) : \alpha + \frac{8\sqrt{2}-6}{6} \le x \le \alpha + \frac{9\sqrt{2}-6}{6}\right\}\Bigg\}\end{aligned} \tag{6}$$

When $\alpha = 0$ and $\omega = 1$, we get figure 1 and when $\alpha = (12-9\sqrt{2})/6$ and $\omega = (18-9\sqrt{2})/6$, we get figure 2

6.2. **Example of §5.2.1, step 2.** . Suppose:

(1) $A = \mathbb{R}$
(2) When defining $f : A \to \mathbb{R}$: i.e.,

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \le x < 0.5 \\ 0.5 & 0.5 \le x \end{cases} \tag{7}$$

(3) $(G_r^\star)_{r\in\mathbb{N}} = (\{(x, f(x)) : -r \le x \le r\})_{r\in\mathbb{N}}$
(4) $G_1^\star = \{(x, f(x)) : -1 \le x \le 1\}$
(5) $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$, using eq. 4 and fig. 1, which is *approximately*

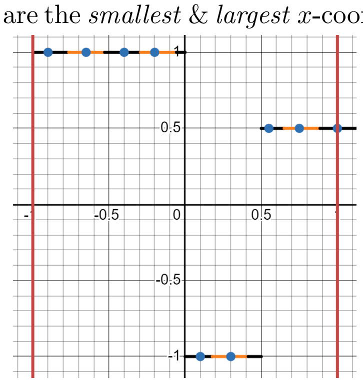
$$\begin{aligned}\{ &\{(x,f(x)) : -1 \le x < -.764\}, \{(x,f(x)) : -.764 \le x < -.528\}, \{(x,f(x)) : -.528 \le x < -.293\} \\ &\{(x,f(x)) : -.293 \le x < -.057\}, \{(x,f(x)) : -.057 \le x < .178\}, \{(x,f(x)) : .178 \le x < .414\} \\ &\{(x,f(x)) : .414 \le x < .65\}, \{(x,f(x)) : .65 \le x < .886\}, \{(x,f(x)) : .886 \le x \le 1.121\}\}\end{aligned} \tag{8}$$

Then, an example of $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{9}$$

Below, we illustrate the sample: i.e., the set of all blue points *in each orange and black line of* $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$ covering $G_1^\star$:

FIGURE 3. The blue points are the "sample points", the alternative black and orange lines are the "covers", and the red lines are the *smallest* & *largest* x-coordinate of $G_1^\star$.



Note, there are multiple samples that can be taken, as long as one sample point is taken from each cover in $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$.

6.3. **Example of §5.2.1, step 3.** Suppose

(1) $A = \mathbb{R}$
(2) When defining $f : A \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & 0 \le x < 0.5 \\ 0.5 & 0.5 \le x \end{cases} \tag{10}$$

(3) $(G_r^\star)_{r\in\mathbb{N}} = (\{(x, f(x)) : -r \le x \le r\})_{r\in\mathbb{N}}$
(4) $G_1^\star = \{(x, f(x)) : -1 \le x \le 1\}$
(5) $\mathbf{C}(\sqrt{2}/6, G_1^\star, 1)$, using eq. 4 and fig. 1, is approx.

$$\begin{aligned}\{\,&\{(x, f(x)) : -1 \le x < -.764\}, \{(x, f(x)) : -.764 \le x < -.528\}, \{(x, f(x)) : -.528 \le x < -.293\} \\ &\{(x, f(x)) : -.293 \le x < -.057\}, \{(x, f(x)) : -.057 \le x < .178\}, \{(x, f(x)) : .178 \le x < .414\} \\ &\{(x, f(x)) : .414 \le x < .65\}, \{(x, f(x)) : .65 \le x < .886\}, \{(x, f(x)) : .886 \le x \le 1.121\}\,\}\end{aligned} \tag{11}$$

(6) $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$, using eq. 9, is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{12}$$

Therefore, consider the following process:

6.3.1. *Step 3a.* If $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ is:

$$\{(-.9, 1), (-.65, 1), (-.4, 1), (-.2, 1), (.1, -1), (.3, -1), (.55, .5), (.75, .5), (1, .5)\} \tag{13}$$

suppose $x_0 = (-.9, 1)$. Note, the following:

(1) $x_1 = (-.65, 1)$ is the next point in the "pathway" since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ with the smallest 2-d Euclidean distance to x_0 instead of x_0.
(2) $x_2 = (-.4, 1)$ is the third point since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ with the smallest 2-d Euclidean distance to x_1 instead of x_0 and x_1.
(3) $x_3 = (-.2, 1)$ is the fourth point since it's a point in $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ with the smallest 2-d Euclidean distance to x_2 instead of x_0, x_1, and x_2.
(4) we continue this process, where the "**pathway**" of $\mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)$ is:

$$(-.9, 1) \to (-.65, 1) \to (-.4, 1) \to (-.2, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1) \tag{14}$$

Note 1. *If more than one point has the minimum 2-d Euclidean distance from x_0, x_1, x_2, etc. take all potential pathways: e.g., using the sample in eq. 13, if $x_0 = (-.65, 1)$, then since $(-.9, 1)$ and $(-.4, 1)$ have the smallest Euclidean distance to $(-.65, 1)$, take **two** pathways:*

$$(-.65, 1) \to (-.9, 1) \to (-.4, 1) \to (-.2, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1)$$

and also:

$$(-.65, 1) \to (-.4, 1) \to (-.2, 1) \to (-.9, 1) \to (.55, .5) \to (.75, .5) \to (1, .5) \to (.3, -1) \to (.1, -1)$$

6.3.2. *Step 3b.* Next, take the length of all line segments in each pathway. In other words, suppose $d(P, Q)$ is the 2-d Euclidean distance between points $P, Q \in \mathbb{R}^2$. Using the pathway in eq. 14, we want:

$$\begin{aligned}\{&d((-.9, 1), (-.65, 1)), d((-.65, 1), (-.4, 1)), d((-.4, 1), (-.2, 1)), d((-.2, 1), (.55, .5)), \\ &d((.55, .5), (.75, .5)), d((.75, .5), (1, .5)), d((1, .5), (.3, -1)), d((.3, -1), (.1, -1))\}\end{aligned} \tag{15}$$

Whose distances can be approximated as:

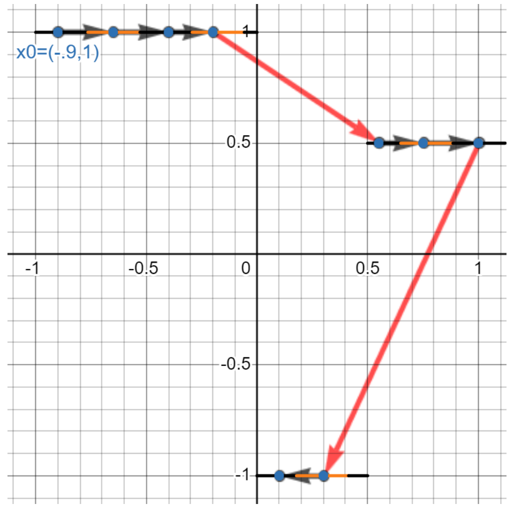
$$\{.25, .25, .2, .901389, .2, .25, 1.655295, .2\}$$

Also, we see the outliers are .901389 and 1.655295 (i.e., notice that the outliers are more prominent for $\varepsilon \ll \sqrt{2}/6$). Therefore, remove .901389 and 1.655295 from our set of lengths:

$$\{.25, .25, .2, .2, .25, .2\}$$

This is illustrated using:

FIGURE 4. The black arrows are the "pathways" whose lengths aren't outliers. The length of the red arrows in the pathway are outliers.



Hence, when $x_0 = (-.9, 1)$, using §5.2.1 step 3b & eq. 13, we note:

$$\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)) = \{.25, .25, .2, .2, .25, .2\} \tag{16}$$

6.3.3. *Step 3c.* To convert the set of distances in eq. 16 into a probability distribution, we take:

$$\sum_{x\in\{.25,.25,.2,.2,.25,.2\}} x = .25 + .25 + .2 + .2 + .25 + .2 = 1.35 \tag{17}$$

Then divide each element in $\{.25, .25, .2, .2, .25, .2\}$ by 1.35

$$\{.25/(1.35), .25/(1.35), .2/(1.35), .2/(1.35), .25/(1.35), .2/(1.35)\}$$

which gives us the probability distribution:

$$\{5/27, 5/27, 4/27, 4/27, 5/27, 4/27\}$$

Hence,

$$\mathbb{P}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) = \{5/27, 5/27, 4/27, 4/27, 5/27, 4/27\} \tag{18}$$

6.3.4. *Step 3d.* Take the shannon entropy of eq. 18:

$$\begin{aligned}
&\mathrm{E}(\mathbb{P}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)))) = \\
&\sum_{x\in\mathbb{P}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^\star,1),1)))} -x\log_2 x = \sum_{x\in\{5/27,5/27,4/27,4/27,5/27,4/27\}} -x\log_2 x = \\
&-(5/27)\log_2(5/27) - (5/27)\log_2(5/27) - (4/27)\log_2(4/27) - (4/27)\log_2(4/27) - (5/27)\log_2(5/27) - (4/27)\log_2(5/27) = \\
&-(15/27)\log_2(5/27) - (12/27)\log_2(4/27) \approx 2.57604
\end{aligned}$$

We shorten $\mathrm{E}(\mathbb{P}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))))$ to $\mathrm{E}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1)))$, giving us:

$$\mathrm{E}(\mathcal{L}((-.9,1), \mathcal{S}(\mathbf{C}(\sqrt{2}/6, G_1^\star, 1), 1))) \approx 2.57604 \tag{19}$$

6.3.5. *Step 3e.* Take the entropy, w.r.t all pathways, of the sample:

$$\{(-.9,1),(-.65,1),(-.4,1),(-.2,1),(.1,-1),(.3,-1),(.55,.5),(.75,.5),(1,.5)\} \tag{20}$$

In other words, we'll compute:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) = \sup_{x_0\in\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1)} \mathrm{E}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1)))$$

We do this by repeating §6.3.1-§6.3.4 for different $x_0 \in \mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1)))$ (i.e., in the equation with multiple values, see note 1)

$$\mathrm{E}(\mathcal{L}((-.9,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 2.57604 \tag{21}$$
$$\mathrm{E}(\mathcal{L}((-.65,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 2.3131, 2.377604 \tag{22}$$
$$\mathrm{E}(\mathcal{L}((-.4,1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 2.3131 \tag{23}$$
$$\mathrm{E}(\mathcal{L}((-.2,-1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 2.57604 \tag{24}$$
$$\mathrm{E}(\mathcal{L}((-.1,-1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 1.86094 \tag{25}$$
$$\mathrm{E}(\mathcal{L}((-.3,-1),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 1.85289 \tag{26}$$
$$\mathrm{E}(\mathcal{L}((.55,.5),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 2.08327 \tag{27}$$
$$\mathrm{E}(\mathcal{L}((.75,.5),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 2.31185 \tag{28}$$
$$\mathrm{E}(\mathcal{L}((1,.5),\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 2.2622 \tag{29}$$

Hence, since the largest value out of eq. 21-29 is 2.57604:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) = \sup_{x_0\in\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1)} \mathrm{E}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\sqrt{2}/6,G_1^{\star},1),1))) \approx 2.57604$$

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Independent Researcher, bharathk98@gmail.com
Email address: `bharathk98@gmail.com`