(5) *Permutation groups and dihedral groups*

Let n be a positive integer greater than 2. The dihedral group D_n is the symmetry group of a regular n-sided polygon centred at the origin. It is generated by a rotation, r, counter-clockwise about the origin through an angle $2\pi/n$, and a reflection s, in an axis running through one of the polygon's vertices and the origin. These generators for D_n are subject to the relations $r^n = e$, $s^2 = e$ and $sr = r^{-1}s$. Each element of D_n can be expressed in the standard form $r^i s^j$, where $0 \leq i \leq n-1$ and $j = 0$ or 1.

(a) For which values of n is the alternating group A_n abelian and for which n is it nonabelian? Justify your answer.

[6]

(b) For which values of n is the dihedral group D_n abelian and for which n is it nonabelian? Justify your answer.

[6]

(c) The **centre** of a group G is

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}.$$

What is the centre of D_n?

[4]

(6) *Subgroups of dihedral groups*

A group G is *Lagrangian* if for every positive divisor d of $|G|$ there is a subgroup H of G with $|H| = d$.

(a) Give a complete description of the subgroups of D_n for all $n \geq 2$. Your description should detail the elements of each subgroup, prove that they are subgroups and prove that there are no other subgroups apart form the ones you describe.

[12]

(b) Explain how your treatment confirms the fact that the dihedral groups are Lagrangian.

[4]

(c) Use the `.subgroups()` and .order() methods of Sage to determine the number and orders of all the subgroups of a few examples of D_n in order to validate the work you've done in part (a).

[4]

(d) Prove which of the subgroups are normal in D_n?

[4]

(7) *The importance of normal subgroups and factor groups*

Normal subgroups and the factor group construction provide a way to get a simplified view of a group G by partitioning its elements into subsets and looking at the operation induced on the partition by the operation from G. In this question you will prove that normal subgroups are the only such way to obtain simplified views of G.

A *congruence* on a group G is an equivalence $\sim$ on G that is compatible with the group operation of G, in the sense that, if $g_1 \sim g_2$ and $h_1 \sim h_2$ then $g_1 h_1 \sim g_2 h_2$.

Let $\sim$ be a congruence on G.

(a) Prove that if $g_1 \sim g_2$ then $g_1^{-1} \sim g_2^{-1}$.

[3]

(b) Prove that the partition of G induced by the equivalence classes of $\sim$ is the partition of G into the cosets of a certain normal subgroup of G.

[7]

(3) Consider the sequence of positive integers a_n, for $n \geq 1$, defined by

$$a_n = 10^{(2^n)} + 1.$$

(a) Prove that the elements of this sequence are pairwise coprime, i.e. prove that if $m \neq n$ then $\gcd(a_m, a_n) = 1$.

[9]

(b) Show how this result, combined with the Fundamental Theorem of Arithmetic, provides another proof that there are an infinite number of primes.

[3]

Hint: *Begin the first part by proving that* $a_n | (a_{n+1} - 2)$ *and then try to extend this divisibility result in a useful way.*

(4) Consider a general arithmetic sequence $x_j = y + jn$, $(j \geq 1)$. Prove that if p is a prime number such that $p \not| n$ then there is some element from the sequence $\{x_n\}_{j=1}^{\infty}$ that is divisible by p.

Hint: *To prove this result you will need to consider the divisibility of the sequence elements using the concept of the congruence relation and modular arithmetic on the integers.*

Your proof of this result should give you a method which, for a given arithmetic sequence and prime, actually allows you to calculate a point in the sequence from which the divisibility property holds. Illustrate your method by calculating the first element from this sequence that is divisible by p, where

$$p = 150000001 = 1.5 \times 10^8 + 1,$$

n is the integer represented by your ID number and $y = 2024$.

[6]