Consider the magnetic field caused by a magnetic monopole that is placed at the origin $\mathcal{O}$, expressed in terms of spherical coordinates as

$$\begin{aligned} \mathbf{B}\colon \mathbb{R}^3 \setminus \{\mathcal{O}\} &\longrightarrow \mathbb{R}^3 \\ (R, \varphi, \theta) &\longmapsto \frac{g}{R^2}\widehat{\mathbf{R}}, \end{aligned}$$

where $\widehat{\mathbf{R}}$ denotes the unit vector in the radial direction. We denote the unit vector in the φ-direction by $\widehat{\boldsymbol{\varphi}}$ and in the θ-direction by $\widehat{\boldsymbol{\theta}}$. We denote the positive z-axis by $\mathcal{D}_+$ and the negative z-axis by $\mathcal{D}_-$ (both subsets include the origin $\mathcal{O}$).

(A) Show that

$$\begin{aligned}\mathbf{A}_+ : \mathbb{R}^3 \setminus \{\mathcal{D}_-\} &\longrightarrow \mathbb{R}^3 \\ (R, \varphi, \theta) &\longmapsto \frac{g(1-\cos(\varphi))}{R\sin(\varphi)}\widehat{\boldsymbol{\theta}}\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}_- : \mathbb{R}^3 \setminus \{\mathcal{D}_+\} &\longrightarrow \mathbb{R}^3 \\ (R, \varphi, \theta) &\longmapsto -\frac{g(1+\cos(\varphi))}{R\sin(\varphi)}\widehat{\boldsymbol{\theta}}\end{aligned}$$

are both vector potentials for $\mathbf{B}$ on their respective domains[1]. ☞ 4/30

Nota bene: You can use the fact that the curl of a vector field $\mathbf{F} = F_R\widehat{\mathbf{R}} + F_\varphi\widehat{\boldsymbol{\varphi}} + F_\theta\widehat{\boldsymbol{\theta}}$ is given by

$$\begin{aligned}\vec{\nabla} \times \mathbf{F} = &\frac{1}{R\sin(\varphi)}\left(\partial_\varphi(F_\theta \sin(\varphi)) - \partial_\theta(F_\varphi)\right)\widehat{\mathbf{R}} + \frac{1}{R}\left(\frac{1}{\sin(\varphi)}\partial_\theta(F_R) - \partial_R(RF_\theta)\right)\widehat{\boldsymbol{\varphi}} + \\ &\frac{1}{R}\left(\partial_R(RF_\varphi) - \partial_\varphi(F_R)\right)\widehat{\boldsymbol{\theta}}\end{aligned}$$

in terms of spherical coordinates.

(B) Show that $\mathbf{B}$ has no vector potential defined on the whole of $\mathbb{R}^3 \setminus \{\mathcal{O}\}$ by calculating the flux of $\mathbf{B}$ through a sphere with fixed radius R that is centred at the origin on the one hand, and Stokes' theorem on the other hand. ☞ 9/30

(C) Find a scalar potential Λ for $\mathbf{A}_+ - \mathbf{A}_-$ on the overlap of the domains of $\mathbf{A}_+$ and $\mathbf{A}_-$. ☞ 5/30

Nota bene: You can use the fact that the gradient of a scalar field f is given by

$$\vec{\nabla} f = \partial_R(f)\widehat{\mathbf{R}} + \frac{1}{R}\partial_\varphi(f)\widehat{\boldsymbol{\varphi}} + \frac{1}{R\sin(\varphi)}\partial_\theta(f)\widehat{\boldsymbol{\theta}}$$

in terms of spherical coordinates.

(D) In quantum mechanics, the wave function ψ of an electrically charged particle with charge e that is moving through this magnetic field can be understood with the vector potential and satisfies the relation

$$\psi_{\mathbf{A}_-} = e^{ie\Lambda}\psi_{\mathbf{A}_+}$$

on the overlap of the domains of $\mathbf{A}_+$ and $\mathbf{A}_-$. Show, by considering the value of $\psi_{\mathbf{A}_-}$ at $\theta = 0$ and $\theta = 2\pi$, that this implies that $\psi_{\mathbf{A}_-}$ is well-defined if and only if

$$2eg \in \mathbb{Z}.$$

☞ 8/30

The above condition is *Dirac's quantisation condition*. In words, it means that electric charged is quantised if and only if magnetic charge is quantised. Because we know that electric charge is quantised (i.e. the charge of an electric particle can only be an integer multiple of e), this means that *if* we find a magnetically charged particle, its charge will be an integer multiples of some constant (assuming that the theory of the electron is correct).

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[1] While, strictly speaking, these expressions are both not defined whenever $\varphi \in \{0, \pi\}$, the limit of $\mathbf{A}_+$ as $\varphi \to 0$ does exist, and the limit of $\mathbf{A}_-$ as $\varphi \to \pi$ also exists, so we only have to leave out $\mathcal{D}_-$ and $\mathcal{D}_+$, respectively.