1. A CHARGED SPHERICAL SHELL ✍ 25/100

Let $a, b \in \mathbb{R}$ be such that $b > a > 0$, and consider the spheres

$$\mathbb{S}_a^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a^2\} \quad \wedge \quad \mathbb{S}_b^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = b^2\}.$$

Denote by $\mathcal{S}_{ab}$ the spherical shell that lies outside $\mathbb{S}_a^2$ but inside $\mathbb{S}_b^2$. Then $\mathcal{S}_{ab}$ can be parametrised using spherical coordinates, with $R \in [a, b]$, $\theta \in [0, 2\pi)$, and $\varphi \in [0, \pi]$. Denote, for any $P \in \mathcal{S}_{ab}$, the distance from P to the origin by R_P, the sphere of radius R_P that is centred at the origin by $\mathbb{S}_{R_P}^2$, and the spherical shell that lies outside $\mathbb{S}_a^2$ but inside $\mathbb{S}_{R_P}^2$ by $\mathcal{S}_{aR_P}$. See *figure* 1.

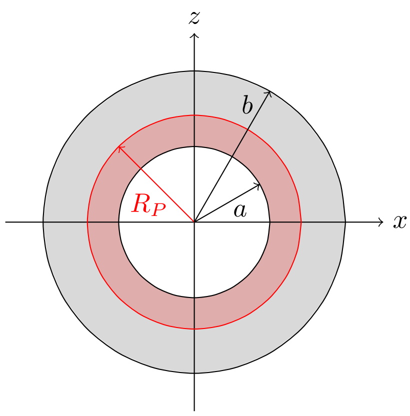

Figure 1: The intersection of the spheres with the xz-plane. The spherical shell $\mathcal{S}_{aR_P}$ is coloured red, and the spherical shell $\mathcal{S}_{ab}$ is the gray region together with the red region.

The region $\mathcal{S}_{ab}$ is electrically charged. The charge density is

$$\begin{aligned} \rho\colon \mathcal{S}_{ab} &\longrightarrow \mathbb{R} \\ P &\longmapsto \frac{1}{R_P^2}. \end{aligned}$$

The electric field inside this region is then given by

$$\begin{aligned} \mathbf{E}\colon \mathcal{S}_{ab} &\longrightarrow \mathbb{R}^3 \\ P &\longmapsto \frac{R_P - a}{R_P^2}\mathbf{R}_P, \end{aligned}$$

where $\mathbf{R}_P$ is a unit vector in the outward radial direction at the point P. With

$$\begin{aligned} \mathrm{E}_{\text{radial}}\colon [a,b] &\longrightarrow \mathbb{R} \\ R &\longmapsto \frac{R-a}{R^2} \end{aligned}$$

the radial component of the electric field, we have $\mathbf{E}(P) = \mathrm{E}_{\text{radial}}(R_P)\mathbf{R}_P$ for each $P \in \mathcal{S}_{ab}$.

(A) Show that

$$\iint_{\mathbb{S}^2_{R_P}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{A} = 4\pi(R_P - a)$$

holds for any $P \in \mathcal{S}_{ab}$, where $\mathrm{d}\boldsymbol{A} = \mathrm{d}A\,\mathbf{R}_P$ for $P \in \mathcal{S}_{ab}$.

(B) Show that

$$\iiint_{\mathcal{S}_{aR_P}} \rho\,\mathrm{d}V = 4\pi(R_P - a).$$

holds for any $P \in \mathcal{S}_{ab}$.

(C) Show that $\vec{\nabla} \cdot \mathbf{E} = \rho$.

(D) Conclude from the above that

$$\begin{aligned} \iint_{\mathbb{S}^2_{R_P}} \mathbf{E} \cdot \mathrm{d}\mathbf{A} &= \iiint_{\mathcal{S}_{aR_P}} \rho\,\mathrm{d}V \\ &= \iiint_{\mathcal{S}_{aR_P}} (\vec{\nabla} \cdot E)\,\mathrm{d}V \end{aligned}$$

holds for any $P \in \mathcal{S}_{ab}$. Give an interpretation of this result.

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2. Extreme values ✍ 25/100

Consider the function

$$\begin{aligned} f\colon \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto (x^2 + 2y^2)e^{-x^2-y^2}. \end{aligned}$$

(A) Determine the extreme values of f.

(B) Determine the extreme values of f on the circle with radius 2 in $\mathbb{R}^2$ that is centred at the origin.

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3. Setting things straight ✍ 20/100

The subset D of $\mathbb{R}^2$ is the one bounded by the four lines defined by the equations $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$.

(A) Sketch D.

(B) Use a suitable coordinate transformation to calculate $\iint_D y^2 \, \mathrm{d}A$.

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4. Domains of integration ✍ 20/100

Consider the double integral

$$\int_0^8 \left(\int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1} \, \mathrm{d}y \right) \mathrm{d}x.$$

(A) For a suitable subset D of $\mathbb{R}^2$, this integral is equal to

$$\iint\limits_D \frac{1}{y^4+1} \, \mathrm{d}A.$$

Give a definition of D and sketch it.

(B) The integral as such is quite difficult. Use your definition of D to rewrite the order of integration and calculate the integral.
