Laplace Transforms and Transfer Functions

1. Determine the Laplace Transform of the following equations:

   a) $f(t) = 3\exp(-4t)$

   b) $f(t) = 5\exp(2t)$

   c) $f(t) = 2\exp(-4(t+2))$

   d) $f(t) = 2\mathrm{t} * \exp(-t) + 4\sin(5t)$

   e) $f(t) = 4 + 4t + 4t^2$

   f) $f(t) = 2\exp(-3t) * \cos(7t)$

[11 marks]

2. Determine the inverse Laplace Transform of the following equations:

a) $F(s) = \frac{1}{2s+1}$

b) $F(s) = \frac{1}{s^2} - \frac{4}{s-2} + \frac{1}{s^2+1}$

c) $F(s) = \frac{3}{s(s+2)}$

d) $F(s) = \frac{4}{(s+2)^2} + \frac{4}{s} + \frac{s}{s^2+2}$

e) $F(s) = 10$

f) $F(s) = \frac{1}{(s+3)} e^{-2s}$

[12 marks]

3) Determine the transfer functions for the following ordinary differential equations:

a) $4\frac{dy(t)}{dt} + 2y(t) = x(t) - 3$

b) $5\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 7x(t) + 4$

c) $3\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 2y(t) = x(t)$-10

d) $4\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + y(t) = 6\frac{dx(t)}{dt} + x(t) - 6$

e) $\frac{d^4y(t)}{dt^4} - 3\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} - 3\frac{dy(t)}{dt} + 6y(t)$

$$= \frac{d^2x(t)}{dt^2} - 4\frac{dx(t)}{dt} + 2x(t) + 5$$

[7 marks]

4) Determine the roots of the following transfer functions. Having determined the value of the roots, state whether their expected response to a step change in the input is stable or unstable, monotonic or oscillatory.

a) $G(s) = \frac{2}{3s+2}$

b) $G(s) = \frac{5}{1-2s}$

c) $G(s) = \frac{5}{(s+2)(s+3)}$

d) $G(s) = \frac{2}{s}$

e) $G(s) = \frac{3}{(4s^2+4s+2)}$

f) $G(s) = \frac{6}{(s^2-4)}$

[13 marks]

5) Rewrite the following transfer functions in the form $\frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$.

Determine the values of τ and ζ. State what kind of response you expect from a step change in the input - overdamped, critically damped, underdamped, undamped sustained oscillations, unstable growing oscillations or runaway.

a) $G(s) = \frac{5}{(s+2)(s+3)}$

b) $G(s) = \frac{3}{(4s^2 - 3s + 2)}$

c) $G(s) = \frac{7}{(s^2 + 2s + 1)}$

d) $G(s) = \frac{8}{(3s^2 + 2s + 1)}$

e) $G(s) = \frac{5}{(s^2 + 4)}$

[15 marks]

6) Figure 1 shows the change in the temperature of water coming out of a boiler following a step change in the water pressure of 0.5 bar at t = 0.

a) Name the type of oscillation that can be seen.

[2 marks]

b) See section 2.7 in the course notes of topic 3. Determine:

i. the gain K (in ^{o}C/bar)
ii. the dead time Δt (in min)
iii. the period of oscillation T (in min)
iv. the frequency ω (in rad/min)
v. the decay ratio *C/B (-)*
vi. the overshoot *B/A (-)*
vii. the rise time (in min)
viii. the settling time (in min)

[8 marks]

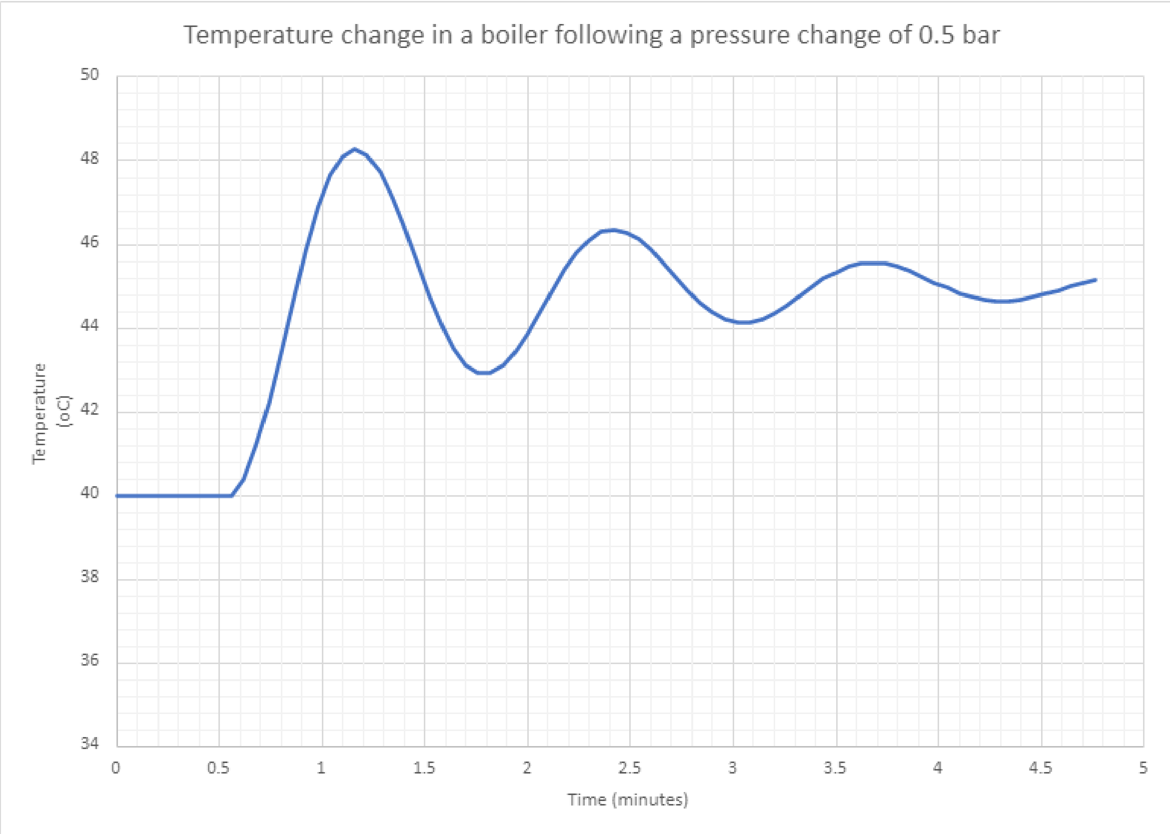

Figure 1.

7) A system described by a step transfer function $G(t) = Ku(t)$ is subjected to a step change at is input $X(t) = \Delta xu(t)$. Show that the output response *Y*(*t*) can be described by a ramp function.

[6 marks]

End of coursework

Process Modelling Data Sheet

The Laplace transforms of an equation $f(t)$ is defined as:

$$ƛ\left[f(t)\right] = F(s) = \int_0^\infty f(t)\exp(-st)\,dt$$

Table 1. Laplace transform of common functions:

Function $f(t)$	Laplace transform $F(s) = ƛ\left[f(t)\right]$
Dirac function $\delta(t)$	1
Unit step $u(t)$	$\frac{1}{s}$
a	$\frac{a}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$\exp(-at)$	$\frac{1}{s+a}$
$t\exp(-at)$	$\frac{1}{(s+a)^2}$
$t^n\exp(-at)$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\exp(-at)\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$\exp(-at)\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 2. First order system

Transfer function	$Y(s) = \left[\frac{K}{\tau s+1}\right]X(s)$

Table 3. Second order system

Transfer function	$Y(s) = \left[\frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}\right]X(s)$
Roots	$r_1 = \frac{-\zeta - \sqrt{\zeta^2 - 1}}{\tau}$; $r_2 = \frac{-\zeta + \sqrt{\zeta^2 - 1}}{\tau}$

Damping ratio value	**Response**
$\zeta \geq 1$	Overdamped= monotonic and stable
$\zeta = 1$	Critically damped; monotonic and stable
$0 \leq \zeta \leq 1$	Underdamped=oscillatory and stable
$\zeta = 0$	Undamped=sustained oscillations
$-1 \leq \zeta \leq 0$	Unstable; growing oscillations
$\zeta \leq -1$	Runaway; monotonic unstable

Underdamped step response	
$r_1 = \rho + i\omega; r_2 = \rho - i\omega$	
Period of oscillation	$T = 2\pi/\omega$
Rise time	= *T*/4
Decay ratio	$= \exp(-(\xi/\tau)T)$
Overshoot	$= \exp(-(\xi/\tau)T/2)$
Settling time	$= 5\tau/\xi$