Question 2

a) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with $E(X_n) = 0$ and $Y_n = \sum_{k=1}^{n} X_{k-1}X_k$. Prove that $(Y_n)_{n\in\mathbb{N}}$ is a martingale.

b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed random variables with $E(X_n) = 0$ and $Var(X_n) = \sigma^2$. Prove that $(Y_n)_{n\in\mathbb{N}}$ where $Y_n = \sum_{k=1}^{n} X_k^2 - n\sigma^2$ is a martingale.

Question 3 Let $(B_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be a Brownian motion and a Poisson process with parameter λ respectively. Let also $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with mean μ and variance σ^2. Assume that the processes $(B_t)_{t\geq 0}$, $(N_t)_{t\geq 0}$ and $(X_n)_{n\in\mathbb{N}}$ to be independent each other and defined on a common stochastic basis $(\Omega, \mathcal{A}, P, \mathcal{F})$.

a) Write the concept of independent increments for a process defined on the same stochastic basis.

b) Prove that $Y_t = B_t + \sum_{k=1}^{N_t} X_k - \lambda\mu t$ is a martingale with respect to $\mathcal{F}$.

Question 4

a) Use the Ito formula to prove that $E(B_t^4) = 3t^2$ (Hint: Use $f(x) = x^4$, apply Ito formula and take expected value on both sides).

Question 5

a) Use the Ito formula to solve the the Ornstein-Ulenbeck stochastic differential equation:

$$dX_t = -\theta X_t dt + \sigma dB_t$$

Hint: Take $f(x, y) = xexp(\theta y)$ and the process $Y_t = (X_t, t)$.