Invariants under Group Actions to Amaze Your Friends

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Source: *Mathematics Magazine*, Dec., 1999, Vol. 72, No. 5 (Dec., 1999), pp. 383-387

Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America

Stable URL: https://www.jstor.org/stable/2690795

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Acknowledgment. This work was supported in part by NSF grant DUE 9450731. I first heard of the problem through Bill Gosper (via Dick Askey). The idea in the previous section was inspired by something one of my high school students, Jan Nelson, did a long time ago. My high school *teacher*, Frank Kelley, just celebrated his 71st birthday, and he's still inspiring young people to study mathematics.

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Invariants Under Group Actions to Amaze Your Friends

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Introduction This paper presents some simple magic tricks that work by themselves based upon mathematical principles. There are whole books devoted to this subject (see, e.g., [2], [3], or [5]), but this article specifically explores card effects that exploit invariants under the (group) action of mixing the cards. Brent Morris' recent book [4] is a wonderful exposition of the mathematics behind some of the effects that take advantage of the groups generated by perfect shuffles. This article orthogonally explores tricks where a *spectator* is allowed to mix the cards. The underlying theme is that a large permutation group leaves an audience with the feeling that the cards are being mixed while leaving an interesting set of invariants under the group action that can be used to perform a (hopefully) startling effect.

TV magic This trick was performed on a recent television program: A volunteer from the audience was handed the four aces from a deck of cards while the performer turned his back. (The reader may wish to take four playing cards, one from each suit, and follow along.) The magician then gave the following instructions:

1. Stack the four cards face-up with the heart at the bottom, then the club, then the diamond, and finally the spade.
2. Turn the spade (the uppermost card) face down.
3. Perform any of the following operations as many times and in any order that you wish:
   (a) Cut any number of cards from the top to the bottom.
   (b) Turn the top two cards over as one.
   (c) Either turn the entire stack over or do not—your choice.
4. Turn the topmost card over, then turn the top two cards over as one, and then turn the top three cards over as one.

At this point, the prestidigitator correctly divines that the club is the only card facing the opposite way from the others. As long as the audience member correctly followed the above directions, the magician is sure to be right.

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Fair play and an easier trick This trick is somewhat surprising because it seems to involve what we will refer to as "fair play." That is, what appears to one person to be mixing cards is actually preserving all of the properties that a second person cares about. Hence the spectator feels that the process is fair, which makes the outcome surprising. Another simple trick that illustrates this is one of the first card tricks that any child learns.

> The magician has a spectator choose a card, memorize it, and return it to the top of the deck. He then allows the spectator to cut the cards as many time as she would like. The magician spreads the cards face up and announces the chosen card.

This works because under the *action* of cutting the cards, the adjacency of pairs of cards is *invariant*. Thus, by remembering the name of the bottom card on the deck (which is secretly glimpsed while the spectator is looking at her own card), the magician can spot the chosen card as the one in front of the secret card in the face-up spread.

The group in this case is the subgroup H of S_{52} generated by the cyclic-shift permutation

$$\sigma = (1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad 51 \quad 52)$$

and the invariant is the set A of adjacent pairs of cards in the deck (where the top and bottom cards are considered an adjacent pair). It is clear that A is not changed by any action in H, so since H appears to the spectator to be mixing the cards but the invariant set A can be used to find a chosen card, this is an effective magic trick.

Analysis of the original problem We next address how to represent the state of the cards for the original trick in this paper. We have a strange sort of permutation where each value has an orientation (face-up versus face-down) as well as a position in the deck. We will represent such a permutation as a "colored" permutation of $1, 2, 3, 4$ with the convention that underlined type represents "face down" for the cards. A typical set of decisions in the trick might go as follows, where the deck starts off face up with a heart (4) at the bottom, then a club (3), then a diamond (2), and then a spade (1).

The original deck is represented as		$1, 2, 3, 4$
(i)	Turning the spade (the uppermost card) face down gives us	$\underline{1}, 2, 3, 4$
(ii)	Cutting *two* cards from the top to the bottom gives us	$3, 4, \underline{1}, 2$
(iii)	Turning the top two cards over as one yields	$\underline{4}, \underline{3}, \underline{1}, 2$
(iv)	Cutting *three* cards from the top to the bottom makes this	$2, \underline{4}, \underline{3}, \underline{1}$
(v)	Turning the top two cards over as one again gives us	$4, \underline{2}, \underline{3}, \underline{1}$
(vi)	Turning the entire stack over yields	$1, 3, 2, \underline{4}$
(vii)	Turning the topmost card over,	$\underline{1}, 3, 2, \underline{4}$
	then the top two cards over as one,	$\underline{3}, 1, 2, \underline{4}$
	and then the top three cards over as one respectively yields finally	$\underline{2}, \underline{1}, 3, \underline{4}$

Note that in the final arrangement Card 3 (the club) is turned up while the other three cards are turned down, so the trick does work with these choices... as if there were ever any doubt. This trick is interesting mathematically in that it really seems that every permutation of the cards could be achieved using the steps for mixing, but clearly this is not so. There are $2^4 \cdot 4! = 384$ ways we can arrange $1, 2, 3, 4$ and

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underline each number or not, but the decisions at Step 3 in the trick will prevent many of these from happening. So the first question is to characterize the outcomes we can and cannot get.

Formally we could represent the permutations that act on the cards as the subgroup H of S_8 (thinking of the four cards' front-and-back pairs as eight objects) generated by the permutations that constitute the three operations in the trick that require decisions, namely cutting one card, turning two cards as one, and flipping over the entire packet. These will be denoted κ, τ, and ϕ respectively and are given below. We will continue to use our more visual notation of letters in two styles to express these mappings. Here the underline merely means that the particular card was reversed from its original orientation.

$$\kappa = ABCD \mapsto BCDA \quad \tau = ABCD \mapsto \underline{BA}CD \quad \phi = ABCD \mapsto \underline{DCBA}$$

The first proposition states in essence that at the end of each step in the trick there will always be 1 or 3 face down cards. Let C_0 denote the set of arrangements of the cards that have 1 or 3 face down cards. Notice that the first step of the trick causes the packet to be in C_0. With four cards in hand it is easy to check that each of the three mappings generating H leaves C_0 fixed. That is,

PROPOSITION 1. *C_0 is invariant under the action of H.*

The next observation is that, in the end, it does not matter which cards are faced up or down, just that the club (3) is faced differently than the others. It does not require a lot of experimentation to realize that sometimes the club is the only card faced down and sometimes it is the only card faced up. Hence, we change our representation to only underline the card (singular by the preceding proposition) that is faced differently in the deck, and the above process looks like this. The deck starts off face up with a heart (4) at the bottom, then a club (3), then a diamond (2), and then a spade (1).

The original deck is represented as		$1, 2, 3, 4$
(i)	Turning the uppermost card face down gives us	$\underline{1}, 2, 3, 4$
(ii)	Cutting *two* cards gives us	$3, 4, \underline{1}, 2$
(iii)	Turning the top two cards over as one yields	$4, 3, 1, \underline{2}$
(iv)	Cutting *three* of cards makes this	$\underline{2}, 4, 3, 1$
(v)	Turning the top two cards over as one again gives us	$\underline{4}, 2, 3, 1$
(vi)	Turning the entire stack over yields	$1, 3, 2, \underline{4}$
(vii)	Turning the topmost card over, then the top two cards over as one, and then the top three cards over as one yields finally	$2, 1, \underline{3}, 4$

Using this representation, we now realize that there are $4 \cdot 4! = 96$ outcomes, although we can still never generate this many of them with the decisions that the spectator is allowed to make. This representation also allows us to make clear the next proposition regarding invariance. Let C_1 denote the arrangements of the packet of cards so that the number 3 card is two cards away from the wrong-way (underlined) card.

PROPOSITION 2. *C_1 is invariant under the action of H.*

Proof. A packet of cards p in C_1 must originally look like one of the following, where here an underlined letter indicates a card reversed from the rest of the packet:

$$3, A, \underline{B}, C \quad C, 3, A, \underline{B} \quad \underline{B}, C, 3, A \quad A, \underline{B}, C, 3$$

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The following table shows what happens to each of the mappings κ, τ, and ϕ act on a packet p.

p	$\kappa(p)$	$\tau(p)$	$\phi(p)$
$3, A, \underline{B}, C$	$A, \underline{B}, C, 3$	$A, 3, B, \underline{C}$	$C, \underline{B}, A, 3$
$C, 3, A, \underline{B}$	$3, A, \underline{B}, C$	$3, C, \underline{A}, B$	$\underline{B}, A, 3, C$
$\underline{B}, C, 3, A$	$C, 3, A, \underline{B}$	$\underline{C}, B, 3, A$	$A, 3, C, \underline{B}$
$A, \underline{B}, C, 3$	$\underline{B}, C, 3, A$	$B, \underline{A}, C, 3$	$3, C, \underline{B}, A$

In every case, the property that defines C_1 *is preserved.*

The last step in the trick is to turn over the topmost card, then the top two cards (as one), and then the top three cards (as one). To represent this action we will use the notation $\mapsto$ to express the execution of each of the three steps. We will use an asterisk to denote that the card has changed its orientation. This final operation on the entire packet is then represented

$$\begin{aligned} ABCD &\mapsto A^*BCD \\ &\mapsto B^*ACD \\ &\mapsto C^*A^*BD \end{aligned}$$

PROPOSITION 3. *If the packet starts with the club two places away from the wrong-way card, then the club will be the wrong-way card after the final operation.*

Proof. Since the original packet starts off with the club two away from the wrong-way card, then we need to consider just the cases where (1) A or C is the club, or (2) B or D is the club. In case (1), the operations above will reverse both the club and the wrong-way card resulting in the club being the wrong-way card. In case (2), the operations will reverse only the two cards that are neither the club nor the wrong-way card resulting in the club being the wrong-way card.

A solitaire game Here is a simple game of solitaire that can be played with a deck of cards in your hands—we used to play a version of this game on car trips since it does not require a table top. It is equivalent to the game "Even Up," which was recently analyzed in [**1**]. The deck is held face up and fanned through with the player removing pairs of cards of the same color whenever they occur adjacent in the deck. Of course, the removal of adjacent pairs may create other adjacent pairs which will also have to be removed. The game ends when there are no more adjacent same-colored pairs to remove. Winning the game means having no cards left at the end. This can be turned into a magic trick as follows:

> The magician calls upon two spectators to each take half the deck of cards and shuffle them independently. They then merge their stacks (alternatively dealing the cards into a single stack) and play the solitaire game. To the amazement of everyone, they find they have won.

The only real trick is that the magician really does split the deck *in half*, meaning that not only does each spectator get 26 cards but also each spectator gets 13 red cards and 13 black cards. If each spectator has 13 red cards and 13 black cards, then they can shuffle their respective halves until they are blue in the face and when they merge them, the solitaire game will be guaranteed to be a winner.

To see why this is true, imagine one spectator's cards came from a blue-backed deck while the other's cards came from a green-backed deck. The final deck will alternate colors of their backs even though the faces of the cards are fairly shuffled.

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Let us decide that the green-backed cards assume the even positions while the blue-backed cards are in the odd positions. As the solitaire game is played, cards are removed from the deck in adjacent pairs which share the same face-color. Hence at any point in the game, (i) the number of blue-backed red cards is equal to the number of green-backed red cards, (ii) the number of blue-backed black cards is equal to the number of green-backed black cards, and (iii) the blue-backed and green-backed cards alternate. Given these three properties that remain invariant as the game is played, it is impossible that the game should ever end in a loss. This is true because a losing final position must consist of cards alternating in face colors, and property (iii) then dictates that the red cards and black cards should have different back colors contrary to properties (i) and (ii).

In terms of invariants under group actions, the permutation group H at work here is the (large!) subgroup of S_{52} generated by permutations that shuffle the odd positions among themselves and the even positions among themselves. Letting C denote the decks of cards that will lead to a solitaire win, we can state the conclusion of the previous discussion as follows.

PROPOSITION 4. *C is invariant under the action of H.*

The reason that this establishes that the trick will work is that the obvious winning arrangement p_0 which has all red cards in the top half of the deck and all black cards in the bottom half of the deck is in C. As a computational aside, this analysis also tells us that the probability of winning this solitaire game with a fairly shuffled deck of cards is simply the probability that the deck is in an order obtainable as in the magic trick. This probability is

$$\frac{\binom{26}{13}^2}{\binom{52}{26}} \approx 0.218 .$$

Unfortunately, this probability is a bit high for this trick to be really amazing since people familiar with the game could decide that the magician was just lucky. It is the sort of trick that would be more effective if done with several pairs of spectators simultaneously.

Conclusions Many magic tricks, particularly those using cards or those involving mentalism, use a set of procedures to make the spectator feel he is making free choices, when in reality the results of these choices are all equivalent for the magician's purposes.

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