Use exact answers for all questions. No numerical approximations allowed (such as decimals).

1. **(12 points)** 🕮 Sections 1.1–1.3

   Write the following complex number in the form $a + bi$ and also in **polar form**. Write exact answers (no decimals), including all arguments.

   a. $\frac{3}{i} + \frac{i}{3}$

   b. $(1+i)(1-\sqrt{3}i)(\sqrt{3}+i)$

   c. $\frac{1+2i}{3-4i}$

   d. A complex number with argument $\frac{\pi}{12}$ and real part 1

2. **(8 points)** 🕮 Section 1.4

   Using the binomial formula in Exercise 1 of Unit 1 and the Formula (1.4-2) on page 29 of Section 1.4 to compute $(2-i)^4$ in two different ways (using polar form taking powers and direct computation using binomial expansion). Compare your results obtained from the two methods. Numerical calculations will not earn marks, all values must be exact (no calculators are needed).

3. **(8 points)** 🕮 Section 1.4

   a. Prove that $(\bar{z})^k = \overline{(z^k)}$, $z \neq 0$ for all $k \in \mathbb{Z}$ and all $z \in \mathbb{C}$. (***Hint:*** $z = re^{i\theta}$.)

   b. Use the previous part to show that if z_0 is a root of the equation $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$, where all the coefficients are real numbers, then $\bar{z}_0$ is also a solution.

4. **(9 points)** 🕮 Section 1.3

   Indicate whether each of the statements below is true or false. If a statement is true, prove it; if a statement is false, give a counterexample.

   a. $\text{Arg}\, z_1 z_2 = \text{Arg} z_1 + \text{Arg} z_2$, $z_1, z_2 \neq 0$.

   b. $\arg \bar{z} = -\arg z$, if z is not a real number.

   c. $\arg(z_1/z_2) = \arg z_1 - \arg z_2$, $z_1, z_2 \neq 0$.

5. **(8 points)** 📖 Section 1.4

   a. Use mathematical induction to prove that for any $n \geq 1$,

   $$1 + z + z^2 + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

   b. Use part (*a*) and De Moivre's formula to establish the following identity

   $$1 + \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{\theta}{2}\right)}$$

6. **(16 points)** 📖 Section 1.4

   a. Where is the function $f(z) = e^{x^2-y^2}[\cos(2xy) - i\sin(2xy)]$ differentiable?

   b. Write $f(z)$ in terms of z (no x, y should appear) and calculate the derivative at all point(s) where it is differentiable.

   c. Let $g(z) = e^{z^2} = e^{x^2-y^2}\cos(2xy) + i[e^{x^2-y^2}\sin(2xy)]$ (it is optional to verify this). Calculate u_x and v_x, where $g(z) = u + iv$. Note that $g(z)$ is an entire function from the expression $g(z) = e^2$.

   d. Use $g'(z) = u_x + iv_x$ to evaluate $g'(i)$.

7. **(4 points)** 📖 Sections 2.3–2.4

   Let $f(z) = e^{2x} + ie^{y}$. Find all values of z where the function is differentiable.

8. **(10 points)** 📖 Sections 2.1–2.3

   a. Let $f(z) = (z^9 - 3iz + \pi i)^{-2081}$, calculate $f'(z)$ at all points where it is continuous (no need to find the domain) and then calculate $\lim\limits_{z\to\infty} f'(z)$.

   b. Let $f(z) = \dfrac{(2z+1)^2}{(z^2+iz+1)^3}$, calculate $\lim\limits_{z\to\frac{\sqrt{5}-1}{2}i} f(z)$ and then calculate $f'(z)$ at all points where it is differentiable.

9. **(4 points)** Section 2.4

   We have a statement "*If $f(z)$ is not an entire function, then $g(z) = f^2(z)$ cannot be an entire function.*". Two students and a professor had the following conversation:

   A: The statement is false. I can give a counter-example. Let $f(z) = \sqrt{z}$, assuming that it is a branch that takes -1 to i. Then it is not analytic on the branch cut, but $g(z) = f^2(z) = z$ is obviously an entire function.

   B: I think this is not a counter-example, so we cannot conclude the statement is false yet.

   Professor: Good! B is correct. The statement is false. Now I give you an assignment: (a) Show that student A's example does not work as a counter-example. (b) Give a (correct) counter-example.

   Now you are asked to complete this short assignment posted by the professor.

10. **(3 points)** Section 2.4

    If both $f(z)$ and $-\overline{f(z)}$ are analytic, what can you say about $f(z)$? Prove your claim.

11. **(15 points)** Sections 2.3–2.4

    Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain in the complex plane.

    a. Show that the family of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal. More precisely, at any point of intersection z_0 of the two curves, the tangent lines to the two curves are perpendicular, if $f'(z_0) \neq 0$. (***Hint:*** Compute the gradient vectors of u and v and use the Cauchy-Riemann equations.)

    b. Let $f(z) = z^2$. Show that at $z = 0$ ($f'(0) = 0$), in order for the two sets of curves have intersection that passing through $z = 0$, $c_1 = c_2 = 0$ is the only possibility. Explain why the above result fails. (***Hint:*** Don't forget the curves must pass through $z = 0$ in order to make the condition $f'(z_0) \neq 0$ in the previous part invalid.)

12. **(3 points)** Section 2.5

    Let $u(x, y) = x^2 - y^2 - 2x + 1$. Prove that it is harmonic and find a harmonic conjugate $v(x, y)$ so that $f(z) = u(x, y) + iv(x, y)$ is analytic.