Throughout this exam, let N denote the final (rightmost) digit of your UCD student number.

1. (a) (i) Determine the natural domain of the function

$$f(x) := \frac{\sqrt{x}}{\sqrt{5-x}}.$$

For full marks, write down the natural domain using interval notation.

(ii) Let f above have codomain $\mathbb{R}$. Is f surjective? Justify your answer. [5]

(b) Determine whether or not the following limits exist and, if so, evaluate them.

(i) $\lim_{x\to 0} \frac{x}{\sin x}$ (ii) $\lim_{x\to 3} \frac{x^2-3}{x-3}$ (iii) $\lim_{x\to 0} \frac{\sqrt{x+N+2}-\sqrt{N+2}}{x}$.

You can state without justification any results from the lecture notes about limits involving trigonometric functions. [8]

(c) Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 3e^{x-2} & x < 2 \\ -3 & x = 2 \\ \dfrac{x^2-x-2}{x-2} & x > 2. \end{cases}$$

Justify your answers to the following questions.

(i) Does $\lim_{x\to 2} g(x)$ exist?

(ii) Is g continuous at 2? [5]

2. (a) Differentiate $f(x) := \dfrac{1}{x-2}$ from **First Principles**. [4]

(b) Differentiate the following functions with respect to x.

(i) $x^{N+3}\log x$ (ii) $\dfrac{e^{2x}}{\cos x}$ (iii) $e^{\sin x}$. [6]

(c) Consider the curve $x^2 + 2xy - y^2 + x = -1$.

(i) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

(ii) Find the equation of the tangent line to the curve at $(1, -1)$. [5]

(d) The function $g : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) and satisfies $g'(x) > 0$ for all $x \in (a, b)$. Use relevant results from the lecture notes to explain why g is injective. [2]

3. Parts (a) – (c) are related. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) \; = \; 5 + x^3 + y^3 - 3xy.$$

(a) Find all first- and second-order partial derivatives of f. [5]

(b) Find the two critical points of f. [4]

(c) Classify both critical points of f. You can use without proof any relevant results from the lecture notes. [4]

(d) Find the linearisation of $g(x) := \sqrt{x} + \sqrt[3]{x}$ at $a = 64$. [3]

4. (a) Determine the following integrals.

(i) $\displaystyle\int x^{-\frac{1}{3}} + \cos x \, dx$ (ii) $\displaystyle\int_1^2 x e^{-(N+10)x} \, dx$

(iii) $\displaystyle\int \cos^{N+2020} x \sin x \, dx$ (iv) $\displaystyle\int_3^7 \frac{\log x}{x} \, dx.$

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(b) (i) Evaluate $\displaystyle\int_0^1 e^x \, dx.$

(ii) Use Simpson's rule with $n = 4$ subintervals to numerically evaluate the integral in 4(b)(i).

(iii) Find the maximum theoretical error in your calculation and verify that this is greater than the actual error.

In question 4(b) you can use numerical approximations obtained from a calculator.

[7]