Question:

Use the variation of parameter method to find a general solution for the variable coefficient equation

$$(\cos x)y(x) = (\cos x)y''(x) + 2(\sin x)y'(x), \ \ 0 < x < 1.$$

Variation of Parameters: $y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$.

If y_1 and y_2 are two linearly independent solutions to the corresponding homogeneous equation, then a particular solution to the nonhomogeneous equation is

$$y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)\ ,$$

where v_1' and v_2' are determined by the equations

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= f(t)/a. \end{aligned}$$

4.6 Variation of Parameters

We have seen that the method of undetermined coefficients is a simple procedure for determining a particular solution when the equation has constant coefficients and the nonhomogeneous term is of a special type. Here we present a more general method, called **variation of parameters,**† for finding a particular solution.

†*Historical Footnote:* The method of variation of parameters was invented by Joseph Lagrange in 1774.

Consider the nonhomogeneous linear second-order equation

(1)

$$ay'' + by' + cy = f(t)$$

and let $\{y_1(t),\ y_2(t)\}$ be two linearly independent solutions for the corresponding homogeneous equation

$$ay'' + by' + cy = 0.$$

Then we know that a general solution to this homogeneous equation is given by

(2)

$$y_h(t) = c_1y_1(t) + c_2y_2(t)\ ,$$

where c_1 and c_2 are constants. To find a particular solution to the nonhomogeneous equation, the strategy of variation of parameters is to replace the constants in (2) by functions of *t*. That is, we seek a solution of (1) of the form†

†In **Exercises 2.3**, **Problem 36**, we developed this approach for first-order linear equations. Because of the similarity of **equations (2)** and **(3)**, this technique is sometimes known as "variation of constants."

(3)

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)\ .$$

Because we have introduced two unknown functions, $v_1(t)$ and $v_2(t)$, it is reasonable to expect that we can impose two equations (requirements) on these functions. Naturally, one of these equations should come from (1). Let's therefore plug $y_p(t)$ given by (3) into (1). To accomplish this, we must first compute $y_p'(t)$ and $y_p''(t)$. From (3) we obtain

$$y_p' = (v_1'y_1 + v_2'y_2) + (v_1y_1' + v_2y_2')\ .$$

To simplify the computation and to avoid second-order derivatives for the unknowns v_1, v_2 in the expression for y_p'', we impose the requirement

(4)

$$v_1'y_1 + v_2'y_2 = 0\ .$$

Thus, the formula for y_p' becomes

(5)

$$y_p' = v_1y_1' + v_2y_2',$$

and so

(6)

$$y_p'' = v_1'y_1' + v_1y''_1 + v_2'y_2' + v_2y_2''\ .$$

Now, substituting y_p, y'_p, and y''_p, as given in (3), (5), and (6), into (1), we find

(7)

$$\begin{aligned} f &= a y''_p + b y'_p + c y_p \\ &= a\left(v'_1 y'_1 + v_1 y''_1 + v'_2 y'_2 + v_2 y''_2\right) + b\left(v_1 y'_1 + v_2 y'_2\right) + c\left(v_1 y_1 + v_2 y_2\right) \\ &= a\left(v'_1 y'_1 + v'_2 y'_2\right) + v_1\left(a y''_1 + b y'_1 + c y_1\right) + v_2\left(a y''_2 + b y'_2 + c y_2\right) \\ &= a\left(v'_1 y'_1 + v'_2 y'_2\right) + 0 + 0 \end{aligned}$$

since y_1 and y_2 are solutions to the homogeneous equation. Thus, (7) reduces to

(8)

$$v'_1 y'_1 + v'_2 y'_2 = \frac{f}{a} .$$

To summarize, if we can find v_1 and v_2 that satisfy both (4) and (8), that is,

(9)

$$\begin{aligned} y_1 v'_1 + y_2 v'_2 &= 0, \\ y'_1 v'_1 + y'_2 v'_2 &= \frac{f}{a}, \end{aligned}$$

then y_p given by (3) will be a particular solution to (1). To determine v_1 and v_2, we first solve the linear system (9) for v'_1 and v'_2. Algebraic manipulation or Cramer's rule (see **Appendix D**) immediately gives

$$v'_1(t) = \frac{-f(t) y_2(t)}{a\left[y_1(t) y'_2(t) - y'_1(t) y_2(t)\right]} \quad \text{and} \quad v'_2(t) = \frac{f(t) y_1(t)}{a\left[y_1(t) y'_2(t) - y'_1(t) y_2(t)\right]},$$

where the bracketed expression in the denominator (the Wronskian) is never zero because of **Lemma 1**, **Section 4.2**. Upon integrating these equations, we finally obtain

(10)

$$v_1(t) = \int \frac{-f(t) y_2(t)}{a\left[y_1(t) y'_2(t) - y'_1(t) y_2(t)\right]}\, dt \quad \text{and} \quad v_2(t) = \int \frac{f(t) y_1(t)}{a\left[y_1(t) y'_2(t) - y'_1(t) y_2(t)\right]}\, dt .$$

Let's review this procedure.

Method of Variation of Parameters

To determine a particular solution to $ay'' + by' + cy = f$:

a. Find two linearly independent solutions $\{y_1(t),\ y_2(t)\}$ to the corresponding homogeneous equation and take

$$y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t) .$$

b. Determine $v_1(t)$ and $v_2(t)$ by solving the system in (9) for $v'_1(t)$ and $v'_2(t)$ and integrating.

c. Substitute $v_1(t)$ and $v_2(t)$ into the expression for $y_p(t)$ to obtain a particular solution.

Of course, in step (b) one could use the formulas in (10), but $v_1(t)$ and $v_2(t)$ are so easy to derive that you are advised not to memorize them.

Example 1

Find a general solution on $(-\pi/2,\ \pi/2)$ to

(11)

$$\frac{d^2y}{dt^2}+y=\tan\ t\,.$$

Solution

Observe that two independent solutions to the homogeneous equation $y''+y=0$ are $\cos t$ and $\sin t$. We now set

(12)

$$y_p(t)=v_1(t)\ \cos\ t+v_2(t)\ \sin\ t$$

and, referring to (9), solve the system

$$\begin{aligned}(\ \cos\ t)v_1'(t)+(\ \sin\ t)v_2'(t)&=0\ \ ,\\ (-\ \sin\ t)v_1'(t)+(\ \cos\ t)v_2'(t)&=\ \tan\ t\,,\end{aligned}$$

for $v_1'(t)$ and $v_2'(t)$. This gives

$$\begin{aligned}v_1'(t)&=-\tan t\ \sin t\,,\\ v_2'(t)&=\tan t\ \cos t=\sin t\,.\end{aligned}$$

Integrating, we obtain

(13)

$$\begin{aligned}v_1(t)\ &=-\int\tan\ t\ \sin\ t\,dt=-\int\frac{\sin^2 t}{\cos t}\,dt\\ &=-\int\frac{1-\cos^2 t}{\cos\ t}dt=\int(\cos\ t-\sec\ t)dt\\ &=\ \sin\ t-\ln|\sec t+\tan t|+C_1\,,\end{aligned}$$

(14)

$$v_2(t)=\int\sin t\,dt=-\cos\ t+C_2.$$

We need only one particular solution, so we take both C_1 and C_2 to be zero for simplicity. Then, substituting $v_1(t)$ and $v_2(t)$ in (12), we obtain

$$y_p(t)=(\sin\ t-\ln|\sec\ t+\ \tan\ t|)\ \cos\ t-\ \cos\ t\ \ \sin\ t\,,$$

which simplifies to

$$y_p(t)=2\,(\cos\ t)\ \ln|\sec\ t+\ \tan\ t|\,.$$

We may drop the absolute value symbols because $\sec t+\ \tan t=(1+\ \sin t)/\cos t>0$ for $-\pi/2<t<\pi/2$.

Recall that a general solution to a nonhomogeneous equation is given by the sum of a general solution to the homogeneous equation and a particular solution. Consequently, a general solution to **equation (11)** on the interval $(-\pi/2,\ \pi/2)$ is

(15)

$$y(t)=c_1\ \cos\ t+c_2\ \sin\ t-(\ \cos\ t)\ \ln(\sec\ t+\ \tan\ t).$$

Note that in the above example the constants C_1 and C_2 appearing in (13) and (14) were chosen to be zero. If we had retained these arbitrary constants, the ultimate effect would be just to add $C_1 \cos t + C_2 \sin t$ to (15), which is clearly redundant.

Example 2

Find a particular solution on $(-\pi/2, \ \pi/2)$ to

(16)

$$\frac{d^2y}{dt^2} + y = \tan t + 3t - 1 \, .$$

Solution

With $f(t) = \tan t + 3t - 1$, the variation of parameters procedure will lead to a solution. But it is simpler in this case to consider separately the equations

(17)

$$\frac{d^2y}{dt^2} + y = \tan t \, ,$$

(18)

$$\frac{d^2y}{dt^2} + y = 3t - 1$$

and then use the superposition principle (**Theorem 3**, page **181**).

In **Example 1** we found that

$$y_q(t) = -(\cos t) \ln(\sec t + \tan t)$$

is a particular solution for **equation (17)**. For **equation (18)** the method of undetermined coefficients can be applied. On seeking a solution to (18) of the form $y_r(t) = At + B$, we quickly obtain

$$y_r(t) = 3t - 1 \, .$$

Finally, we apply the superposition principle to get

$$\begin{aligned} y_p(t) &= y_q(t) + y_r(t) \\ &= -(\cos t) \ln(\sec t + \tan t) + 3t - 1 \end{aligned}$$

as a particular solution for **equation (16)**.

Note that we could not have solved **Example 1** by the method of undetermined coefficients; the nonhomogeneity $\tan t$ is unsuitable. Another important advantage of the method of variation of parameters is its applicability to linear equations whose coefficients a, b, c are functions of t. Indeed, on reviewing the derivation of the system (9) and the **formulas (10)**, one can check that we did not make any use of the constant coefficient property; i.e., the method works provided we know a pair of linearly independent solutions to the corresponding homogeneous equation. We illustrate the method in the next example.

Example 3

Find a particular solution of the variable coefficient linear equation

(19)

$$t^2 y'' - 4ty' + 6y = 4t^3, \quad t > 0 \, ,$$

given that $y_1(t) = t^2$ and $y_2(t) = t^3$ are solutions to the corresponding homogeneous equation.

Solution

The functions t^2 and t^3 are linearly independent solutions to the corresponding homogeneous equation on $(0, \infty)$ (verify this !) and so (19) has a particular solution of the form

$$y_p(t) = v_1(t)t^2 + v_2(t)t^3 .$$

To determine the unknown functions v_1 and v_2, we solve the system (9) with $f(t) = 4t^3$ and $a = a(t) = t^2$:

$$\begin{aligned} t^2 v_1'(t) + t^3 v_2'(t) &= 0 \\ 2t v_1'(t) + 3t^2 v_2'(t) &= f/a = 4t . \end{aligned}$$

The solutions are readily found to be $v_1'(t) = -4$ and $v_2'(t) = 4/t$, which gives $v_1(t) = -4t$ and $v_2(t) = 4$ ln t. Consequently,

$$y_p(t) = (-4t)t^2 + (4 \ \ln t)t^3 = 4t^3(-1 + \ln t)$$

is a solution to (19).

Variable coefficient linear equations will be discussed in more detail in the next section.

4.7 Variable-Coefficient Equations

The techniques of **Sections 4.2** and **4.3** have explicitly demonstrated that solutions to a linear homogeneous constant-coefficient differential equation,

(1)

$$ay'' + by' + cy = 0 \ ,$$

are defined and satisfy the equation over the whole interval $(-\infty, \ +\infty)$. After all, such solutions are combinations of exponentials, sinusoids, and polynomials.

The variation of parameters formula of **Section 4.6** extended this to nonhomogeneous constant-coefficient problems,

(2)

$$ay'' + by' + cy = f\ (t) \ ,$$

yielding solutions valid over all intervals where $f\ (t)$ *is continuous* (ensuring that the integrals in (10) of **Section 4.6** containing $f\ (t)$ exist and are differentiable). We could hardly hope for more; indeed, it is debatable what *meaning* the differential **equation (2)** would have at a point where $f\ (t)$ is undefined, or discontinuous.

Therefore, when we move to the realm of equations with *variable* coefficients of the form

(3)

$$a_2\ (t)y'' + a_1\ (t)y' + a_0\ (t)y = f\ (t) \ ,$$

the most we can expect is that there are solutions that are valid over intervals where all four "governing" functions—$a_2(t)$, $a_1(t)$, $a_0(t)$, and $f(t)$—are continuous. Fortunately, this expectation is fulfilled except for an important technical requirement—namely, that the coefficient function $a_2(t)$ must be nonzero over the interval.[†]

†Indeed, the whole nature of the equation—reduction from *second*-order to *first*-order—changes at points where $a_2(t)$ is zero.

Typically, one divides by the nonzero coefficient $a_2(t)$ and expresses the theorem for the equation in **standard form** [see (4), below] as follows.

Existence and Uniqueness of Solutions

Theorem 5

If $p(t)$, $q(t)$, and g(t) are continuous on an interval (a, b) that contains the point t_0, then for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (*a*, *b*) to the initial value problem

(4)

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)\,; \quad y(t_0) = Y_0\,, \quad y'(t_0) = Y_1\,.$$

Example 1

Determine the largest interval for which **Theorem 5** ensures the existence and uniqueness of a solution to the initial value problem

(5)

$$(t-3)\frac{d^2y}{dt^2} + \frac{dy}{dt} + \sqrt{t}\,y = \ln\,t\,; \quad y(1) = 3\,, \quad y'(1) = -5\,.$$

Solution

The data $p(t)$, $q(t)$, and $g(t)$ in the standard form of the equation,

$$y''+py'+qy = \frac{d^2y}{dt^2} + \frac{1}{(t-3)}\frac{dy}{dt} + \frac{\sqrt{t}}{(t-3)}\,y = \frac{\ln t}{(t-3)} = g\,,$$

are simultaneously continuous in the intervals $0 < t < 3$ and $3 < t < \infty$. The former contains the point $t_0 = 1$, where the initial conditions are specified, so **Theorem 5** guarantees (5) has a unique solution in $0 < t < 3$.

Theorem 5, embracing existence and uniqueness for the variable-coefficient case, is difficult to prove because we can't construct explicit solutions in the general case. So the proof is deferred to Chapter 13.[†] However, it is instructive to examine a special case that we can solve explicitly.

†All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems*, 7th ed.

Cauchy–Euler, or Equidimensional, Equations

Definition 2

A linear second-order equation that can be expressed in the form

(6)

$$at^2y''(t) + bty'(t) + cy = f(t),$$

where a, b, and c are constants, is called a **Cauchy–Euler,** or **equidimensional, equation.**

For example, the differential equation

$$3t^2y'' + 11ty' - 3y = \sin\ t$$

is a Cauchy–Euler equation, whereas

$$2y'' - 3ty' + 11y = 3t - 1$$

is *not* because the coefficient of y'' is 2, which is not a constant times t^2.

The nomenclature *equidimensional* comes about because if *y* has the dimensions of, say, meters and *t* has dimensions of time, then each term t^2y'', ty', and *y* has the same dimensions (meters). The coefficient of $y''(t)$ in (6) is at^2, and it is zero at $t = 0$; equivalently, the standard form

$$y'' + \frac{b}{at}\ y' + \frac{c}{at^2}\ y = \frac{f(t)}{at^2}$$

has discontinuous coefficients at $t = 0$. Therefore, we can expect the solutions to be valid only for $t > 0$ or $t < 0$. Discontinuities in f, of course, will impose further restrictions.

To solve a *homogeneous* Cauchy–Euler equation, for $t > 0$, we exploit the equidimensional feature by looking for solutions of the form $y = t^r$, because then $t^2 y'',\ ty'$, and *y* each have the form *(constant)*×*t^r*:

$$y = t^r, \quad ty' = trt^{r-1} = rt^r, \quad t^2 y'' = t^2 r\,(r-1)t^{r-2} = r\,(r-1)t^r,$$

and substitution into the homogeneous form of (6) (that is, with $g = 0$) yields a simple quadratic equation for *r*:

$$ar\,(r-1)t^r + brt^r + ct^r = \left[ar^2 + (b-a)r + c\right]t^r = 0, \quad \text{or}$$

(7)

$$ar^2 + (b-a)r + c = 0,$$

which we call the associated *characteristic equation*.

Example 2

Find two linearly independent solutions to the equation

$$3t^2 y'' + 11ty' - 3y = 0, \quad t > 0.$$

Solution

Inserting $y = t^r$ yields, according to (7),

$$3r^2 + (11-3)r - 3 = 3r^2 + 8r - 3 = 0\ ,$$

whose roots $r = 1/3$ and $r = -3$ produce the independent solutions

$$y_1\,(t) = t^{1/3}\ , \quad y_2\,(t) = t^{23} \quad (\text{for } t > 0)\ .$$

Clearly, the substitution $y = t^r$ into a homogeneous *equidimensional* equation has the same simplifying effect as the insertion of $y = e^{rt}$ into the homogeneous *constant-coefficient* equation in **Section 4.2**. That means we will have to deal with the same encumbrances:

1. What to do when the roots of (7) are complex
2. What to do when the roots of (7) are equal

If *r* is complex, $r = \alpha + i\beta$, we can interpret $t^{\alpha+i\beta}$ by using the identity $t = e^{\ln t}$ and invoking Euler's formula [**equation (5)**, **Section 4.3**]:

$$t^{\alpha+i\beta} = t^{\alpha}t^{i\beta} = t^{\alpha}e^{i\beta \ln t} = t^{\alpha}\left[\cos\left(\beta \ln t\right) + i \sin\left(\beta \ln t\right)\right].$$

Then we simplify as in **Section 4.3** by taking the real and imaginary parts to form independent solutions:

(8)

$$y_1 = t^{\alpha}\cos\left(\beta \ln t\right), \quad y_2 = t^{\alpha}\sin\left(\beta \ln t\right).$$

If *r* is a double root of the characteristic **equation (7)**, then independent solutions of the Cauchy–Euler equation on $(0, \infty)$ are given by

(9)

$$y_1 = t^r, \quad y_2 = t^r \ln t.$$

This can be verified by direct substitution into the differential equation. Alternatively, the second, linearly independent, solution can be obtained by *reduction of order*, a procedure to be discussed shortly in **Theorem 8**. Furthermore, **Problem 23** demonstrates that the substitution $t = e^x$ changes the homogeneous Cauchy–Euler equation into a homogeneous constant-coefficient equation, and the formats (8) and (9) then follow from our earlier deliberations.

We remark that if a homogeneous Cauchy–Euler equation is to be solved for $t < 0$, then one simply introduces the change of variable $t = -\tau$, where $\tau > 0$. The reader should verify via the chain rule that the identical characteristic **equation (7)** arises when $\tau^r = (-t)^r$ is substituted in the equation. Thus the solutions take the same form as (8), (9), but with *t* replaced by $-t$; for example, if *r* is a double root of (7), we get $(-t)^r$ and $(-t)^r \ln\ (-t)$ as two linearly independent solutions on $(-\infty,\ 0)$.

Example 3

Find a pair of linearly independent solutions to the following Cauchy–Euler equations for $t > 0$.

a. $t^2 y'' + 5ty' + 5y = 0$
b. $t^2 y'' + ty' = 0$

Solution

For part (a), the characteristic equation becomes $r^2 + 4r + 5 = 0$, with the roots $r = -2 \pm i$, and (8) produces the real solutions $t^{-2}\cos(\ln t)$ and $t^{-2}\sin(\ln t)$.

For part (b), the characteristic equation becomes simply $r^2 = 0$ with the double root $r = 0$, and (9) yields the solutions $t^0 = 1$ and ln *t*.

In **Chapter 8** we will see how one can obtain power series expansions for solutions to variable-coefficient equations when the coefficients are *analytic* functions. But, as we said, there is no procedure for explicitly solving the general case. Nonetheless, thanks to the existence/ uniqueness result of **Theorem 5**, most of the other theorems and concepts of the preceding sections are easily extended to the variable-coefficient case, with the proviso that they apply only over intervals in which the governing functions $p(t),\ q(t),\ g(t)$ are continuous. Thus we have the following analog of **Lemma 1**, page **160**.

A Condition for Linear Dependence of Solutions

Lemma 3

If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation

(10)

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

on an interval *I* where the functions $p(t)$ and $q(t)$ are continuous and if the Wronskian†

†The determinant representation of the Wronskian was introduced in **Problem 34**, **Section 4.2**.

$$W[y_1, y_2](t) := y_1(t)y_2'(t) - y_1'(t)y_2(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

is zero at *any* point *t* **of *I***, then y_1 and y_2 are linearly dependent **on *I***.

As in the constant-coefficient case, the Wronskian of two solutions is either identically zero or never zero on *I*, with the latter implying linear independence on *I*.

Precisely as in the proof for the constant-coefficient case, it can be verified that any linear combination $c_1y_1 + c_2y_2$ of solutions y_1 and y_2 to (10) is also a solution. In fact, these are the only solutions to (10) as stated in the following result.

Representation of Solutions to Initial Value Problems

Theorem 6

If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential **equation (10)** that are linearly independent on an interval *I*, then every solution to (10) on *I* is expressible as a linear combination of y_1 and y_2. Moreover, the initial value problem consisting of **equation (10)** and the initial conditions $y(t_0) = Y_0,\ y'(t_0) = Y_1$ has a unique solution on *I* for any point t_0 in *I* and any constants Y_0, Y_1.

As in the constant-coefficient case, the linear combination $y_h = c_1y_1 + c_2y_2$ is called a **general solution** to (10) on *I* if $y_1,\ y_2$ are linearly independent solutions on *I*.

For the nonhomogeneous equation

(11)

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)\ ,$$

a general solution on *I* is given by $y = y_p + y_h$, where $y_h = c_1y_1 + c_2y_2$ is a general solution to the corresponding homogeneous equation (10) **on *I* and y_p is a particular solution to** (11) **on *I***. In other words, the solution to the initial value problem stated in **Theorem 5** must be of this form for a suitable choice of the constants c_1, c_2. This follows, just as before, from a straightforward extension of the superposition principle for variable-coefficient equations described in **Problem 30**.

As illustrated at the end of the **Section 4.6**, if linearly independent solutions to the homogeneous **equation (10)** are known, then y_p can be determined for (11) by the variation of parameters method.

Variation of Parameters

Theorem 7

If y_1 and y_2 are two linearly independent solutions to the homogeneous **equation (10)** on an interval *I* where *p*(*t*), *q*(*t*), and *g*(*t*) are continuous, then a particular solution to (11) is given by $y_p = v_1 y_1 + v_2 y_2$, where v_1 and v_2 are determined up to a constant by the pair of equations

$$y_1 v'_1 + y_2 v'_2 = 0,$$
$$y_1' v_1' + y_2' v_2' = g,$$

which have the solution

(12)

$$v_1(t) = \int \frac{-g(t)\, y_2(t)}{W[y_1, y_2](t)}\, dt\,, \quad v_2(t) = \int \frac{g(t) y_1(t)}{W[y_1, y_2](t)}\, dt\,.$$

Note the **formulation (12)** presumes that the differential equation has been put into standard form [that is, divided by $a_2(t)$].

The proofs of the constant-coefficient versions of these theorems in **Sections 4.2** and **4.5** did not make use of the constant-coefficient property, so one can prove them in the general case by literally copying those proofs but interpreting the coefficients as variables. Unfortunately, however, there is no construction analogous to the method of undetermined coefficients for the variable-coefficient case.

What does all this mean? The only stumbling block for our completely solving nonhomogeneous initial value problems for equations with variable coefficients,

$$y'' + p(t) y' + q(t) y = g(t)\,; \quad y(t_0) = Y_0 \quad , \quad y'(t_0) = Y_1\,,$$

is the lack of an explicit procedure for constructing independent solutions to the associated homogeneous **equation (10)**. If we *had* y_1 and y_2 as described in the variation of parameters formula, we could implement (12) to find y_p, formulate the general solution of (11) as $y_p + c_1 y_1 + c_2 y_2$, and (with the assurance that the Wronskian is nonzero) fit the constants to the initial conditions. But with the exception of the Cauchy–Euler equation and the ponderous power series machinery of **Chapter 8**, we are stymied at the outset; there is no general procedure for finding y_1 and y_2.

Ironically, we only need *one* nontrivial solution to the associated homogeneous equation, thanks to a procedure known as *reduction of order* that constructs a second, linearly independent solution y_2 from a known one y_1. So one might well feel that the following theorem rubs salt into the wound.

Variation of Parameters

In **Section 4.6** we discussed the method of variation of parameters for a general constant-coefficient second-order linear equation. Simply put, the idea is that if a general solution to the homogeneous equation has the form $x_h(t) = c_1 x_1(t) + c_2 x_2(t)$, where $x_1(t)$ and $x_2(t)$ are linearly independent solutions to the homogeneous equation, then a particular solution to the nonhomogeneous equation would have the form $x_p(t) = v_1(t)x_1(t) + v_2(t)x_2(t)$, where $v_1(t)$ and $v_2(t)$ are certain functions of *t*. A similar idea can be used for systems.

Let $\mathbf{X}(t)$ be a fundamental matrix for the homogeneous system

(6)

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t),$$

where now *the entries of* **A** *may be any continuous functions of t*. Because a general solution to **(6)** is given by $\mathbf{X}(t)\mathbf{c}$, where **c** is a constant $n \times 1$ vector, we seek a particular solution to the nonhomogeneous system

(7)

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$$

of the form

(8)

$$\mathbf{x}_p(t) = \mathbf{X}(t)\upsilon(t),$$

where $\upsilon(t) = \text{col}(\upsilon_1(t), \ldots, \upsilon_n(t))$ is a vector function of *t* to be determined.

To derive a formula for $\upsilon(t)$, we first differentiate **(8)** using the matrix version of the product rule to obtain

$$\mathbf{x}_p'(t) = \mathbf{X}(t)\upsilon'(t) + \mathbf{X}'(t)\upsilon(t).$$

Substituting the expressions for $\mathbf{x}_p(t)$ and $\mathbf{x}_p'(t)$ into **(7)** yields

(9)

$$\mathbf{X}(t)\upsilon'(t) + \mathbf{X}'(t)\upsilon(t) = \mathbf{A}(t)\mathbf{X}(t)\upsilon(t) + \mathbf{f}(t).$$

Since $\mathbf{X}(t)$ satisfies the matrix equation $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$, equation **(9)** becomes

$$\begin{aligned}\mathbf{X}\upsilon' + \mathbf{A}\mathbf{X}\upsilon &= \mathbf{A}\mathbf{X}\upsilon + \mathbf{f},\\ \mathbf{X}\upsilon' &= \mathbf{f}.\end{aligned}$$

Multiplying both sides of the last equation by $\mathbf{X}^{-1}(t)$ [which exists since the columns of $\mathbf{X}(t)$ are linearly independent] gives

$$\upsilon'(t) = \mathbf{X}^{-1}(t)\,\mathbf{f}(t).$$

Integrating, we obtain

$$\upsilon(t) = \int \mathbf{X}^{-1}(t)\,\mathbf{f}(t)dt.$$

Hence, a particular solution to **(7)** is

(10)

$$\mathbf{x}_p(t) = \mathbf{X}(t)\upsilon(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\,\mathbf{f}(t)dt.$$

Combining **(10)** with the solution $\mathbf{X}(t)\mathbf{c}$ to the homogeneous system yields the following general solution to **(7)**:

(11)

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\,\mathbf{f}(t)\,(t)dt\,.$$

The elegance of the derivation of the variation of parameters formula **(10)** for systems becomes evident when one compares it with the more lengthy derivations for the scalar case in **Sections 4.6** and **6.4**.

Given an initial value problem of the form

(12)

$$\mathbf{x}\prime(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0\,,$$

we can use the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ to solve for **c** in **(11)**. Expressing $\mathbf{x}(t)$ using a definite integral, we have

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\,\mathbf{f}(s)ds\,.$$

From the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, we find

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \mathbf{X}(t_0)\mathbf{c} + \mathbf{X}(t_0)\int_{t_0}^{t_0} \mathbf{X}^{-1}(s)\,\mathbf{f}(s)ds = \mathbf{X}(t_0)\mathbf{c}\,.$$

Solving for **c**, we have $\mathbf{c} = \mathbf{X}^{-1}(t_0)\mathbf{x}_0$. Thus, the solution to **(12)** is given by the formula

(13)

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{21}(t_0)\mathbf{x}_0 + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\,\mathbf{f}(s)\,ds\,.$$

To apply the variation of parameters formulas, we first must determine a fundamental matrix $\mathbf{X}(t)$ for the homogeneous system. In the case when the coefficient matrix **A** is constant, we have discussed methods for finding $\mathbf{X}(t)$. However, if the entries of **A** depend on *t*, the determination of $\mathbf{X}(t)$ may be extremely difficult (entailing, perhaps, a matrix power series!).