1. Differentiate the following from First Principles.

(a) $f(x) = \frac{1}{x^2}$, $x \neq 0$

(b) $f(x) = \cos x$.

2. Using the tools available in Chapter 3, differentiate the following expressions with respect to x.

(a) $\sin x \tan x$

(b) $\frac{\cos x}{\log x}$

(c) $(x^8 - 3x^3) \log x \sin x$

(d) $\sin^{2022} x$

(e) $e^{x^4 \cos x}$

(f) $-\log(\cos x)$

(g) $\cos(\log x^2)$

(h) $\frac{\cos(\sin x)}{x}$

(i) $\log(\log(\log x))$.

Example 3.4. Find the slope of the tangent line to the function f defined by $f(x) = x^2$ at the point a. In other words, find the derivative of $f(x) = x^2$ at a.

Solution. From Definition 3.3, we know that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

In this example, $f(x) = x^2$, so substituting into the limit formula, we get

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} && \text{Example 1.7} \\ &= \lim_{h \to 0} \frac{2ah + h^2}{h} \\ &= \lim_{h \to 0} 2a + h = 2a. \end{aligned}$$

□

Example 3.5. Differentiate $f(x) = \sin x$ from first principles and in so doing show that

$$f'(x) = \cos x.$$

Solution.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}. \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Lemma 1.33 (3)} \\ &= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \end{aligned}$$

$$= \lim_{h \to 0} \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right)$$

$$= \sin x \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) + \cos x \left(\lim_{h \to 0} \frac{\sin h}{h} \right) \quad \text{Theorem 2.29 (1)}$$

Here we may use Exercise 2.41 (2) and Theorem 2.37:

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1$$

$$= \cos x.$$

□