Assignment 2

Note: This assignment consists of 5 problems of equal weight.

Due: After Unit 9

1. Given an interval $x > 0$ and the function set

   $$S = \{\ln(x^5), \ln 5, 4\ln(x), \ln(5x)\},$$

   construct a linear ODE of lowest order such that each function in S is one of the set's solutions on the interval.

2. Reduce the system of first-order equations given below to a single equation of higher order and solve that equation. Use the method of undetermined coefficients.

   $$\begin{cases} 2y(t) - x'(t) = t^2 \\ x(t) - y'(t) - y(t) = 3 \end{cases}$$

3. Given the equation

   $$L[y(x)] = \frac{1}{x}$$

   in *standard* form (see p. 319 of the textbook), where L is a linear operator of the second order with fundamental solution set $\{x, x^2\}$, find the general solution.

4. Use the method of variation of parameters to solve

   $$\begin{cases} x'(t) + y'(t) + y(t) = \dfrac{\ln t}{t} e^{-2(t+\ln 2)} - t \\ y'(t) - 4x(t) = 2t^2. \end{cases}$$

5. (Transient Current)

   An RLC series circuit has a voltage source given by $E(t) = 20\sin(t)$ V, an inductor of 3 H, a resistor 6 Ω, and a capacitor of $\dfrac{1}{3}$ F. Find the current $I(t)$ in this circuit for $t > 0$ if

   $$I(0) = 2, \quad I'(0) = 0.$$

   Find the moment of time when the transient current is equal to zero.

6.1 Basic Theory of Linear Differential Equations

A *linear* differential equation of order *n* is an equation that can be written in the form

(1)

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) = b(x),$$

where $a_0(x), a_1(x), \cdots, a_n(x)$ and $b(x)$ depend only on *x*, not *y*. When $a_0, a_1, \ldots, a_n$ are all constants, we say equation (1) has **constant coefficients;** otherwise it has **variable coefficients**. If $b(x) \equiv 0$, equation (1) is called **homogeneous;** otherwise it is **nonhomogeneous**.

In developing a basic theory, we assume that $a_0(x),\ a_1(x),\ \cdots,\ a_n(x)$ and $b(x)$ are all continuous on an interval *I* and $a_n(x) \neq 0$ on *I*. Then, on dividing by $a_n(x)$, we can rewrite (1) in the **standard form**

(2)

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x),$$

where the functions $p_1(x), \ldots, p_n(x)$ and $g(x)$ are continuous on *I*.

For a linear higher-order differential equation, the initial value problem always has a unique solution.

Existence and Uniqueness

Theorem 1

Suppose $p_1(x), \ldots, p_n(x)$ and $g(x)$ are each continuous on an interval $(a,\ b)$ that contains the point x_0. Then, for any choice of the initial values $\gamma_0,\ \gamma_1, \ldots,\ \gamma_{n-1}$, there exists a unique solution $y(x)$ on the whole interval (a, b) to the initial value problem

(3)

(3)

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x),$$

(4)

$$y(x_0) = \gamma_0,\ \gamma'(x_0) = \gamma_1, \ldots,\ y^{(n-1)}(x_0) = \gamma_{n-1}\ .$$

The proof of **Theorem 1** can be found in Chapter 13.†

†All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems*, 7th ed.

Example 1

For the initial value problem

(5)

$$x(x-1)y''' - 3xy'' + 6x^2y' - (\cos\ x)y = \sqrt{x+5};$$

(6)

$$y(x_0) = 1, \qquad y'(x_0) = 0, \qquad y''(x_0) = 7,$$

determine the values of x_0 and the intervals $(a,\ b)$ containing x_0 for which **Theorem 1** guarantees the existence of a unique solution on $(a,\ b)$.

Solution

Putting equation (5) in standard form, we find that $p_1(x) = -3/(x-1)$, $p_2(x) = 6x/(x-1)$, $p_3(x) = -(\cos x)/[x(x-1)]$, and $g(x) = \sqrt{x+5}/[x(x-1)]$. Now $p_1(x)$ and $p_2(x)$ are continuous on every interval not containing $x=1$, while $p_3(x)$ is continuous on every interval not containing $x=0$ or $x=1$. The function $g(x)$ is not defined for $x<-5, x=0$ and $x=1$, but is continuous on $(-5,0), (0,1)$ and $(1,\ \infty)$. Hence, the functions p_1, p_2, p_3, and g are *simultaneously* continuous on the intervals $(-5,0)$, $(0,\ 1)$, and $(1,\ \infty)$. From **Theorem 1** it follows that if we choose $x_0 \in (-5,\ 0)$, then there exists a unique solution to the initial value problem (5)–(6) on the whole interval $(-5,\ 0)$. Similarly, for $x_0 \in (0,1)$, there is a unique solution on $(0,\ 1)$ and, for $x_0 \in (1,\ \infty)$, a unique solution on $(1,\ \infty)$. ◆

If we let the left-hand side of equation (3) define the *differential operator* L,

(7)

$$L[y] := \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = \left(D^n + p_1 D^{n-1} + \cdots + p_n\right)[y],$$

then we can express equation (3) in the operator form

(8)

$$L[y](x) = g(x).$$

It is essential to keep in mind that L is a *linear* operator—that is, it satisfies

(9)

$$L[y_1 + y_2 + \cdots + y_m] = L[y_1] + L[y_2] + \cdots + L[y_m],$$

(10)

$$L[cy] = cL[y] \qquad (c \text{ any constant}).$$

These are familiar properties for the differentiation operator D, from which (9) and (10) follow (see **Problem 25**).

As a consequence of this linearity, if $y_1, \ldots, y_m$ are solutions to the *homogeneous* equation

(11)

$$L[y](x) = 0,$$

then any linear combination of these functions, $C_1 y_1 + \cdots + C_m y_m$, is also a solution, because

$$L[C_1 y_1 + C_2 y_2 + \cdots + C_m y_m] = C_1 \cdot 0 + C_2 \cdot 0 + \cdots + C_m \cdot 0 = 0.$$

Imagine now that we have found n solutions $y_1, \ldots, y_n$ to the nth-order linear equation (11). Is it true that *every* solution to (11) can be represented by

(12)

$$C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

for appropriate choices of the constants $C_1, \ldots, C_n$? The answer is yes, provided the solutions $y_1, \ldots, y_n$ satisfy a certain property that we now derive.

Let $\phi(x)$ be a solution to (11) on the interval (a,b) and let x_0 be a fixed number in (a,b). If it is possible to choose the constants $C_1, \ldots, C_n$ so that

(13)

$$\begin{aligned}
C_1 y_1(x_0) &+ \cdots + C_n y_n(x_0) &= \phi(x_0), \\
C_1 y'_1(x_0) &+ \cdots + C_n y'_n(x_0) &= \phi\prime(x_0), \\
\vdots \quad & \qquad \vdots & \vdots \\
C_1 y_t{}^{(n-1)}{}_1(x_0) &+ \cdots + C_n y_n{}^{(n-1)}(x_0) &= \phi^{(n-1)}(x_0),
\end{aligned}$$