Consider the sphere $S^2 \subseteq \mathbb{R}^3$. The (unit) sphere is defined by $\{(x,y,z) \mid x^2+y^2+z^2=1\}$. At any point (x,y,z) on the sphere the outward pointing unit normal is given by $\vec{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (Note $\bar{n}$ is a unit vector because $x^2+y^2+z^2=1$)

Fix a point $(x,y,z) \in S^2$, then the tangent space of S^2 at (x,y,z) consist of all vectors in $\mathbb{R}^3$ that are orthogonal to $\bar{n}=(x,y,z)$.

Find the matrix of the ~~po~~ orthogonal projection from $\mathbb{R}^3$ to the tangent space of S^2. You want to find the matrix of this projection with respect to the standard basis.

Problem: Let $\langle , \rangle$ be the usual inner product on $\mathbb{C}^3$.

Def^{n}: A 3×3 real matrix is called a (proper) rotation if $A^T = A^{-1}$ and $\det(A)=1$ i.e. $\langle Ax,y\rangle = \langle x,A^Ty\rangle = \langle x,A^{-1}y\rangle$.

(a) A 3^{rd} degree polynomal with real coeficents always has at least one real root λ_1. We know there is an eigenvector (also real) for A with respect to that real root.

(a) Use the fact that $A^T = A^{-1}$ to conclude that the value of the real root is 1. If $\bar{u}$ is the corresponding eigenvector the $A\bar{u}=\bar{u}$. (This will be a vector in $\mathbb{R}^3$ that A rotates about.)

(b) If all of the eigenvalues of A are real then A must be the identity matrix. You will need to use $\det(A) = 1$.

(c) Suppose that A has only one real ~~root~~ eigenvalue, the the other two roots of $\det(A - \lambda I)$ must be complex. If λ is one of the complex roots then $\bar{\lambda}$, the complex conjugate of λ, must be the other complex root. Show $\lambda \cdot \bar{\lambda} = 1$.

(d) Let v_1 be an eigenvector for A corresponding to λ. Then $\bar{v}_1$ is an eigenvector for A corresponding to the root $\bar{\lambda}$. Just like complex numbers, we can write $V_1 = Re(V_1) + i\,Im(V_1)$ hence $V_2 = Re(V_1) - i\,Im(V_1)$.

(e) Let H be the subspace of $\mathbb{C}^3$ such that $H^{\perp} = \{u\}$ (recall $Au = u$). Show that $H^{\perp}$ is invariant under A. i.e. if $h \in H$ then $Ah \in H$.

(f) $H^{\perp}$ is a two dimensional complex subspace of $\mathbb{C}^3$. We have $V_1 = W_1 + iW_2$ with W_1 and W_2 real vectors. Let K be the real subspace in $\mathbb{R}^3$ such that $K^{\perp} = \{u\}$. (Here we mean $K^{\perp} = \{r \in \mathbb{R}^3 \mid \langle r, k\rangle = 0 \ \forall k \in K)$

Show that K is invariant under A (recall we started with A being a real matrix)

(g) Since $\lambda\bar{\lambda}=1$ we have if $\lambda = x+iy$ $\lambda\bar{\lambda}=x^2+y^2=1$ so there exist an angle θ with $\cos(\theta)=x$, $\sin(\theta)=y$ i.e. $\lambda=\cos(\theta)+i\sin(\theta)$

Compute $\lambda\cdot\bar{V}_1=\lambda(W_1+iW_2)$ W_1, W_2 real.

Under the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$

i.e. $x+iy \sim \begin{bmatrix} x \\ y \end{bmatrix}$ Show that $\lambda\cdot\bar{V}_1 \sim \begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin\theta & \cos(\theta) \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$

Therefor we have shown that:

let A be a proper rotation in $\mathbb{R}^3$ (so $\det(A)=1$) then $\exists$ a vector $u\in\mathbb{R}^3$ s.t. $Au=u$ and in the plane $K=\{u\}^\perp$ A acts by rotation through an angle θ. (Nothing for you to do here I'm just summerizing the consequence of the problem.

Problem: Let $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be two vectors in $\mathbb{R}^3$

for the moment let $\bar{i}, \bar{j}, \bar{k}$ denote the three standard basis vectors. So any vector, like u, can be written

$u = u_1\bar{i} + u_2\bar{j} + u_3\cdot\bar{k}$

The crossed product $u \times v$ is the vector

$$u \times v = \det \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

where this is short hand for, if we were to expand along the top row,

$$(-1)^{1+1} \cdot \vec{i} \cdot \det \begin{vmatrix} v_2 & v_3 \\ u_2 & u_3 \end{vmatrix} +$$

$$(-1)^{1+2} \cdot \vec{j} \cdot \det \begin{vmatrix} v_1 & v_3 \\ u_1 & u_3 \end{vmatrix} +$$

$$(-1)^{1+3} \cdot k \det \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix}$$

(a) Fix u then $v \to u \times v$ is a linear map in v from $\mathbb{R}^3 \to \mathbb{R}^3$. Find the matrix of this linear map with respect to the standard basis.

(b) We know that matrix mult corresponds to composition of linear maps. Use this fact to show $w \times (u \times v) \neq (w \times u) \times v$

(c) (Triple product) Let $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

Show $\langle \vec{w}, u \times v \rangle = \det \begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$

Conclude that $u \perp u \times v$ and $v \perp u \times v$