Generalizing all Intuitive Means

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August 2021

1 Defining the Hausdorff Measure and Average w.r.t to the Measure

Suppose (X, d) is a metric space and $E \subseteq X$. Let $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an **(exact) dimension function** (or **gauge function**) which is monotonically increasing, strictly positive, and right continuous [1]. If

$$\mu_\delta^h(E) = \inf \left\{ \sum_{i=1}^{\infty} h(\text{diam}(C_i)) : \text{diam}(C_i) \leq \delta, E \subseteq \bigcup_{i=1}^{\infty} C_i \right\} \tag{1}$$

where *diam* is the diameter of a set and:

$$\mu^h(E) = \sup_{\delta > 0} \mu_\delta^h(E) \tag{2}$$

is the *Hausdorff Outer Measure*, we define h so $\mu^h(E)$ is strictly positive and finite for a majority of "nice" sets that are measurable in the sense of Caratheodory [2].

Note meaningful gauge functions don't exist for all "nice" sets. (I'll explain this in detail in the next section).

If $f : A \to \mathbb{R}$, and A is a bounded subset of $\mathbb{R}^d$, the average with respect to the Hausdorff Measure is:

$$m_f(A) := \frac{1}{\mu^h(A)} \int_A f(x) \, d\mu^h \tag{3}$$

and, furthermore; when A is unbounded (with $t \in \mathbb{R}^+$) $m_f(A)$ can be adjusted as:

$$m'_f(A) := \lim_{t \to \infty} \frac{1}{\mu^h(A \cap [-t, t])} \int_{A \cap [-t,t]} f(x) \, d\mu^h \tag{4}$$

where we add $[-t, t]$ so when $A = \mathbb{R}$, the density of positive real numbers is:

$$\frac{\mu^h\left(\mathbb{R}^+ \cap [-t, t]\right)}{\mu^h(\mathbb{R} \cap [-t, t])} = \frac{\mu^h\left((0, t]\right)}{\mu^h\left([-t, t]\right)} = 1/2 \tag{5}$$

the density of negative real numbers is

$$\frac{\mu^h\left(\mathbb{R}^- \cap [-t,t]\right)}{\mu^h(\mathbb{R}\cap[-t,t])} = \frac{\mu^h([-t,0))}{\mu^h([-t,t])} = 1/2 \tag{6}$$

And note this would be intuitive since $[-t,t]$ has a mid-point of zero which is neither positive nor negative.

2 Motivation for Extending the Mean From the Hausdorff Measure

The function $m'_f(A)$ gives a satisfying average that is unique for a majority measurable A in the sense of Caratheodory; however, there are measurable A without meaningful gauge functions since they're either σ-finite with respect to the counting-measure (e.g. Countably-Infinite sets) or their gauge function doesn't exist (e.g. the Liouville Numbers [3]). In these cases, $m'_f(A)$ can't exist since $\mu^h(A)$ is neither positive nor finite.

While there are methods to extending $m'_f(A)$, I haven't found an extension which gives a unique, satisfying average for the ***largest class*** *of measurable A (in the sense of Carathedory).*

One uses non-standard measure theory [4] but the extension isn't unique as it requires ultra-filters, Zorn's Lemma and equivalent principles.

Another extends $m'_f(A)$ to A in the fractal setting ([5],[6]) but does not work for non-fractal, measurable A.

Other options are found in the work of Attila Losonczi (e.g. [7]) where he provides all averages and their properties; however, *I'm unsure if the averages he mentions are defined for nowhere-continuous f which has a domain dense in $\mathbb{R}$ but with no meaningful gauge function.*

For instance, consider $f : \mathbb{Q}\cap[0,1] \to \mathbb{R}$ and

$$f(x) = \begin{cases} 2 & x \in \left\{a^2 : a \in \mathbb{Q}\right\} \cap [0,1] \\ 1 & x \in \left(\mathbb{Q} \setminus \left\{a^2 : a \in \mathbb{Q}\right\}\right) \cap [0,1] \end{cases} \tag{7}$$

In this case, is the average 1, 2 *or a value in between?*

We must choose a unique, satisfying average for this case, since with other cases of A, we choose $m'_f(A)$, or the averages in [5] and [6].

Since I don't fully understand uncountable, measurable sets that have no gauge function I'll focus on defining a unique, satisfying average for f defined on countably infinite subsets of real numbers. (I don't know if what I define will be compatible with $m'_f(A)$, [4] [5] and [6] or have properties Losonczi listed in [9], [10] and [11]).

If the average I define is satisfying I need a generalization of $m'_f(A)$ to equal the average I will define when A is countably infinite.

3 Defining A Satisfying Average Over Countably Infinite Sets

3.1 Current Method

Suppose $f : A \to \mathbb{R}$ and A is a countably infinite, bounded subset of $\mathbb{R}$.

If $t \in \mathbb{N}$ and $\{a_n\}_{n=1}^{\infty}$ is an enumeration of A, the average of f is:

$$\lim_{t\to\infty} \frac{f(a_1) + f(a_2) + \cdots + f(a_t)}{t} \tag{8}$$

but when f is nowhere continuous and A is dense on a subset of $\mathbb{R}$, if more than one enumeration gives a defined average, the overall average could vary from different enumerations.

Therefore, I need a choice function that picks an enumeration which gives a unique, satisfying average; however, I first must generalize the enumeration, redefine the average, then explain why this is necessary.

3.2 Generalizing the Enumeration with Structures

Suppose $F_1, F_2, ...$ are a sequence of finite subsets of A where

1. $F_1 \subset F_2 \subset ...$
2. $\bigcup_{n=1}^{\infty} F_n = A$.

We denote the sequence of subsets as a **structure** of A which we denote $\{F_n\}$ and an example of a structure, such as when $A = \left\{\frac{1}{m} : m \in \mathbb{N}\right\}$, is $\{F_n\}_{n\in\mathbb{N}} = \left\{\left\{\frac{1}{m} : m \in \mathbb{N}, m \leq n\right\}\right\}_{n\in\mathbb{N}}$.

Note F_n generalizes the enumeration since as n increases by one, if $|F_n|$ increases by one, then $\{F_n\}$ behaves as an enumeration.

Further note there are multiple structures of A. Again for $A = \left\{\frac{1}{m} : m \in \mathbb{N}\right\}$, a second structure of the set is

$$\{F_n\}_{n\in\mathbb{N}} = \left\{\left\{\frac{1}{2m} : m \in \mathbb{N}, m \le n\right\} \cup \left\{\frac{1}{2m+1} : m \in \mathbb{N}, m \le 2n\right\}\right\}_{n\in\mathbb{N}}.$$

3.2.1 Equivalent and Non-Equivalent Structures

Furthermore, structures may *equivelant* or *non-equivelant.*

For instance, with structures F_n and F_j, they're *equivalent* if:

$$\exists(n_1 \in \mathbb{N})\exists(j_1 \in \mathbb{N})\forall(n \in \mathbb{N})\forall(j \in \mathbb{N})\left(\left[(n \ge n_1) \wedge (j \ge j_1)\right] \Rightarrow \left[F_n \subseteq F_j \vee F_j \subseteq F_n\right]\right) \tag{9}$$

which means F_n and F_j are *equivelant* if there exists an n_1 and j_1 where for all n greater than n_1, and all j greater than j_1, $F_n \subseteq F_j$ or $F_j \subseteq F_n$

Moreover, with structures F_n and F_j, they are *non-equivalent* if:

$$\forall(n_1 \in \mathbb{N})\forall(j_1 \in \mathbb{N})\exists(n \in \mathbb{N})\exists(j \in \mathbb{N})\left(\left[(n \ge n_1) \wedge (j \ge j_1)\right] \wedge \left[F_n \nsubseteq F_j \wedge F_j \nsubseteq F_n\right]\right) \tag{10}$$

which means F_n and F_j are non-equivalent if no matter how large n_1 and j_1 are, there exists a larger n and j where $F_n \nsubseteq F_j$ and $F_j \nsubseteq F_n$

3.3 Defining Average From Structure

Suppose $f : A \to \mathbb{R}$ and A is a countably infinite subset of the real numbers. Using the rigorous definition of the limit of sequences, there exists an average $\hat{m}_f(\{F_n\}, A)$, such for every arbitrarily small positive ϵ there exists a sufficiently large integer N such for all $n \ge N$.

$$\left|\frac{1}{|F_n|}\sum_{x\in F_n} f(x) - \hat{m}_f(\{F_n\}, A)\right| \le \epsilon \tag{11}$$

I expect that equivalent structures give the same average and non-equivalent structures give different averages.

When there are non-equivalent structures of A, the average varies depending on the structure chosen. Therefore, *I want to find a group of equivelant structures to gives a satisfying, unique average if one structure of A has a defined average.*

3.4 Reasoning behind Redefining the Enumeration

I defined the structures to get a structure $\{F_n\}$ with a greater rate of decrease in discrepancy (see next section) which otherwise can't be achieved with enumerations.

3.5 Current Definition of Discrepancy and Equidistribution

To understand *discrepancy* we first must understand *equidistribution*.

A structure $\{F_n\}$ is **equidistributed** or **uniformly distributed** on $A_t = [\inf(A \cap [-t,t]), \sup(A \cap [-t,t])]$, if for any sub-interval $[c,d]$ of A_t we have:

$$\lim_{t\to\infty} \lim_{n\to\infty} \frac{|F_n \cap [c,d]|}{|F_n|} = \frac{d-c}{\ell(A_t)} \tag{12}$$

where $\ell(A_t)$ is the length of the interval A_t

We add $[-t,t]$ so when A has no infima or suprema, the limit on the left side of equation [12] exists.

Note current measures of **discrepancy** measure the maximum point of density deviation from a uniform or equidistributed sample:

$$\sup_{\inf(A\cap[-t,t])\leq c\leq d\leq \sup(A\cap[-t,t])} \left| \frac{|F_n \cap [c,d]|}{|F_n|} - \frac{d-c}{\ell(A_t)} \right| \tag{13}$$

with more rigorous definitions stated in [12] and [13] (where we replace $\{a_1, ..., a_N\}$ with F_n and N with $|F_n|$). Unfortunately the discrepancy of most structures converges to zero as $n \to \infty$. This makes it impossible to find a structure with a lower discrepancy than the rest.

One solution is finding a $\{F_n\}$ *where the lower bound of its' discrepancy converges to zero the fastest.* Unfortunately, I am not confident with the current measures of discrepancy as most of them *calculate the maximum point of density deviation rather than the overall deviation from an equidistributed structure*).

3.6 Redefining Discrepancy

Below are steps to measuring the *overall deviation* of a structure from an equidistributed structure).

1. Arrange the values in F_n from least to greatest and take the absolute difference between consecutive elements. Call this ΔF_n. (Note ΔF_n is **not a set** since if absolute differences repeat, *we don't delete the repeating differences.*)

1.1 If $A = \left\{ \frac{1}{m} : m \in \mathbb{N} \right\}$ and $\{F_n\}_{n \in \mathbb{N}} = \left\{ \left\{ \frac{1}{m} : m \in \mathbb{N}, m \leq n \right\} \right\}_{n \in \mathbb{N}}$ then $F_4 = \{1, 1/2, 1/3, 1/4\}$

Arranging F_4 from least to greatest gives us $\{1/4, 1/3, 1/2, 1\}$

Therefore, $\Delta F_4 = \{|1/4 - 1/3|, |1/2 - 1/3|, |1/2 - 1|\} = \{1/12, 1/6, 1/2\}$. (None of the differences here are the same, but in the examples below, at least two of the differences are equivalent.)

1.2 If $A = \mathbb{Q} \cap [0, 1]$ and $\{F_n\}_{n \in \mathbb{N}} = \left\{ \left\{ \frac{j}{k} : j, k \in \mathbb{N}, k \leq n, 0 \leq j \leq k \right\} \right\}_{n \in \mathbb{N}}$ then the elements of F_4, arranged from least to greatest is, $F_4 = \{0, 1/4, 1/3, 1/2, 2/3, 3/4, 1\}$ and

$\Delta F_4 = \{|0 - 1/4|, |1/4 - 1/3|, |1/2 - 1/3|, |2/3 - 1/2|, |3/4 - 2/3|, |1 - 3/4|\} =$

$\{1/4, 1/12, 1/6, 1/6, 1/12, 1/4\}$. (Here the difference 1/4 repeats two times but we do not delete the second 1/4)

2. Divide ΔF_t by the sum of all its elements so we get a distribution where all the elements sum to 1. We shall call this $\Delta F_n / \sum_{x \in \Delta F_n} x$ or *the information probability of the structure*

   2.1 From example 1.1 note $\sum_{x \in \Delta F_3} x = 1/2 + 1/6 + 1/12 = 3/4$ and $\Delta F_3 / \sum_{x \in \Delta F_3} x = 4/3 \cdot \{1/2, 1/6, 1/12\} = \{2/3, 2/9, 1/9\}$.

   Note the elements in this set sum to 1 and act as a probability distribution (despite not being actual probabilities)

3. Since the elements of information probability always sum to 1, we can calculate its deviance from a discrete uniform distribution using Entropy which is written as

$$E(F_n) = - \sum_{j \in \Delta F_n / \sum_{x \in \Delta F_n} x} j \log j \tag{14}$$

(Note the smaller the *deviation from a disrete uniform distribution*, the greater the entropy of the information probability and the lower the structure's *discrepancy*. Moreover, if $E(F_n) \to \infty$ as $n \to \infty$, we say $\{F_n\}$ is *equidistributed*).

   3.1 From $\Delta F_3 / \sum_{x \in \Delta F_3} x$, in example 2.1, $E(F_3)$ is the same as

$$-\sum_{j\in\{2/3,2/9,1/9\}} j\log j = -\left(2/3\log(2/3) + 2/9\log(2/9) + 1/9\log(1/9)\right)$$
$$\approx .369$$

3.7 Determining which Structure has a Greater Rate of Decrease in Discrepancy

If two structures of A, which are $\{F_n\}$ and $\{F_j\}$ have the same number of elements, the structure whose entropy has a greater value also has a lower discrepancy.

However, there are structures where for all n and j, they never have the same cardinality. This means finding the one with a lower discrepancy is impossible.

For alternate approaches, we find structures with a greater rate of decrease in discrepancy defined as follows.

If $\{F_n\}$ and $\{F_j\}$ are non-equivalent, $\{F_n\}$ has a **greater** *rate of decrease in discrepancy* than $\{F_j\}$ if:

$$\exists(k\in\mathbb{N})\forall(n\in\mathbb{N})\left(n\geq k \Rightarrow |F_n| < \inf\{|F_j| : j\in\mathbb{N}, E(F_j)\geq E(F_n)\}\right) \tag{15}$$

(Meaning F_j needs a larger cardinality to get the same amount of discrepancy as F_n)

3.8 Determining which Structure has the Greatest Rate of Decrease in Discrepancy

Suppose $\mathbb{S}(A)$ represents the set of all structures of A, then $\{F'_n\}\in\mathbb{S}(A)$ (and equivalent structures) have the **greatest** *rate of decrease in the discrepancy* if:

$$\exists(k\in\mathbb{N})\forall(n\in\mathbb{N})\left(n\geq k\Rightarrow |F'_n| < \inf\left\{|F_j| : j\in\mathbb{N}, F_j\in\mathbb{S}(A), E(F_j)\geq E(F'_n)\right\}\right) \tag{16}$$

(Note F'_n and all F_j must be non-equivelant, and similar to equation [15], all structures F_j need a larger cardinality to get the same amount of discrepancy as F'_n)

(As a final note, with most cases of A, $\{F'_n\}$ does not exist; however, by specifying the structures we wish for, we can adjust equation [16] to get the structure we desire).

3.9 Specifying Structures I Wish To Choose

For a given A, if $\{F''_n\}$ is the structure I want my choice function to choose, then I want:

1. When $p \in \mathbb{N}$, $a, b \in \mathbb{R}$, $a < b$, and $A = \{\sqrt[p]{a} : a \in \mathbb{Q} \cap [a,b]\}$, $\{F''_n\}$ should equal $\left\{\sqrt[p]{m/n!} : m \in \mathbb{Z}, an! \le m \le bn!\right\}$ if $\hat{m}_f(\{F''_n\}, A)$ is defined and finite.

2. When $A = \{1/m : m \in \mathbb{N}\}$ and $[\times]$ is the nearest integer function, $\{F''_n\} = \{1/[2^n/m] : m \in \mathbb{N}, 1 \le m \le 2^n\}$ when $\hat{m}_f(\{F''_n\}, A)$ is defined and finite.

3. When A is almost nowhere dense (e.g. $\left\{\frac{1}{m} : m \in \mathbb{N}\right\}$), $\{F''_n\}$ should be points with the smallest 1-d Euclidean Distance from each point in $C_n = \{m/2^n : a \cdot 2^n \le m \le b \cdot 2^n\}$ (unless the point in C_n is a limit point of A where a minimum distance won't exist).

   Like with other cases, $\hat{m}_f(\{F''_n\}, A)$ must be defined and finite.

The reason for choosing these $\{F''_n\}$ is to get an intuitive average for $\hat{m}_f(\{F_n\}, A)$ (eq:[9]) for nowhere continuous f.

For example, suppose $f : \left\{\frac{1}{m} : m \in \mathbb{N}\right\} \to \mathbb{R}$ and $A = \left\{\frac{1}{m} : m \in \mathbb{N}\right\}$, where $\{F''_n\} = \{1/[2^n/m] : m \in \mathbb{N}, 1 \le m \le 2^n\}$ and

$$f(x) = \begin{cases} 1/\sqrt{x} & x \in \left\{1/(2^j) : j \in \mathbb{N}\right\} \\ 1 & \text{otherwise} \end{cases} \tag{17}$$

If I try a natural $\{F_n\}$ (i.e. $\left\{\frac{1}{m} : m \in \mathbb{N}, m \le n\right\}$), $\hat{m}_f(\{F_n\}, A) = 1$ but the values of $1/\sqrt{x}$, for $x \in \left\{1/2^j : j \in \mathbb{N}\right\}$, are *significantly* larger than 1. Therefore, it may be reasonable that $1/\sqrt{x}$ should have more weight on the average.

As noted earlier when $A = \left\{\frac{1}{m} : m \in \mathbb{N}, m \le n\right\}$, the structure I want is $\{F''_n\} = \{1/[2^n/m] : m \in \mathbb{N}, 1 \le m \le 2^n\}$.

Using a calculator, I found $m_f(\{F''_n\}, A)$ is approximately 2.707107; however, note for $f : \left\{\frac{1}{m} : m \in \mathbb{N}\right\} \to \mathbb{R}$, if we replace $1/\sqrt{x}$ with $1/x$:

$$f(x) = \begin{cases} 1/x & x \in \left\{1/(2^j) : j \in \mathbb{N}\right\} \\ 1 & \text{otherwise} \end{cases} \tag{18}$$

then $\hat{m}_f(\{F''_n\}, A) = \infty$.

Using the choice function in the next section, I believe it's possible to get a finite $\hat{m}_f(\{F''_n\}, A)$ as long as there exists an $\{F_n\}$ where $\hat{m}_f(\{F_n\}, A)$ exists.

3.10 Defining the choice function

The simplest choice function that I assume gives $\{F''_n\}$ I want for bounded A is:

$$\exists(k\in\mathbb{N})\forall(z\in\mathbb{N})\left(z\geq k\Rightarrow\frac{1}{z}\sum_{n=1}^{z}|F''_n|>\frac{1}{z}\sum_{n=1}^{z}\sup\left\{|F_j|:j\in\mathbb{N},\{F_j\}\in\mathbb{S}'(A),\frac{E(F_{j+1})-E(F_j)}{|F_{j+1}|/|F_j|}\geq\frac{E(F''_{n+1})-E(F''_n)}{|F''_{n+1}|/|F''_n|}\right\}\right) \tag{19}$$

where $\mathbb{S}'(A)$ *is the set of all* $\{F_n\}$ *where* $\hat{m}\left(\{F_n\},A\right)$ *exists.*

Unfortunately, I would need several examples to check if the choice function gives the $\{F''_n\}$ I'm looking for. (See the previous section: for instance, if $A=\{\sqrt{a}:a\in\mathbb{Q}\cap[0,1]\}$ does the choice function in [19] give $\{F''_n\}=\left\{\sqrt{m/n!}:m\in\mathbb{N},|m|\leq n!\right\}$)?

For unbounded A; however, the choice function might give infinite possibilities for $\{F''_n\}$. For example, with $A=\{\sqrt{a}:a\in\mathbb{Q}\}$, we might have $\{F_n\}=\left\{\sqrt{m/n!}:m\in\mathbb{N}\right\}\cap[an,bn]$ where $a<b$.

Hence, we restrict A to $A\cap[-t,t]$ and solve $\{F''_n\}$ for every $t\in\mathbb{R}$ which we call $\left\{F''_{n,t}\right\}$. The choice function is:

$$\exists(k\in\mathbb{N})\forall(z\in\mathbb{N})\left(z\geq k\Rightarrow\frac{1}{z}\sum_{n=1}^{z}|F''_{n,t}|>\frac{1}{z}\sum_{n=1}^{z}\sup\left\{|F_{j,t}|:j\in\mathbb{N},\{F_{j,t}\}\in\mathbb{S}'(A\cap[-t,t]),\frac{E(F_{j+1,t})-E(F_{j,t})}{|F_{j+1,t}|/|F_{j,t}|}\geq\frac{E(F''_{n+1,t})-E(F''_{n,t})}{|F''_{n+1,t}|/|F''_{n,t}|}\right\}\right) \tag{20}$$

and hopefully it gives a unique group of equivalent $\{F_{n,t}\}$ when t is finite.

Next, we continue taking $\hat{m}_f(\left\{F''_{n,t}\right\},A\cap[-t,t])$ as t increases to infinity. We call the result $\overline{\mathbf{m}}_f\left(A\right)$ which give a satisfying average for f defined on countably infinite sets.

In more rigorous terms, using the definition of limit of sequences, there exists an average $\overline{\mathbf{m}}_f\left(A\right)$, such for every arbitrarily small positive ϵ there exists a sufficiently large integer N such for all $t\geq N$.

$$\left|\overline{\mathbf{m}}_f\left(A\right)-\hat{m}_f\left(\left\{F''_{n,t}\right\},A\cap[-t,t]\right)\right|\leq\epsilon \tag{21}$$

3.11 Questions on $\overline{\mathbf{m}}_f\left(A\right)$

1. Has an unique, satisfying average on countably infinite sets already been found? Does it give the same results as $\overline{\mathbf{m}}_f\left(A\right)$?

2. From A in Section 3.9, does $\hat{m}_f\left(\left\{F_n''\right\}, A\right)$ give the same average as $\overline{\mathbf{m}}_f\left(A\right)$

3. Are there simpler ways to determine the structure with the greatest rate of decrease in discrepancy such that it replaces entropy and helps to define a simpler choice function which gives the same result as $\overline{\mathbf{m}}_f\left(A\right)$

3.12 Generalized Mean

If $f: A \to \mathbb{R}$, A is a subset of $\mathbb{R}$, $\text{avg}_f(A)$ is a unique satisfying average of f defined on sets measurable in the sense of Caratheodory, then $\text{avg}_f(A)$ is defined as:

$$\text{avg}_f(A) := \begin{cases} m_f'(A) \;\; \text{(See eq: [4])} & A \text{ has a gauge function} \\ \text{Averages in [5], [6]} & A \text{ is fractal but has no gauge function} \\ \overline{\mathbf{m}}_f\left(A\right) & A \text{ is countably infinite, non fractal-like and for} \\ & \text{at least one structure, } \hat{m}_f(\left\{F_n\right\}, A) \text{ is defined} \\ \text{Unknown} & A \text{ is uncountable and non-fractal with} \\ & \text{no gauge function} \\ \text{Undefined} & \text{Satisfying average cannot exist e.g. there is} \\ & \text{no } \left\{F_n\right\} \text{ where } \hat{m}_f(\left\{F_n\right\}, A) \text{ exists} \end{cases} \tag{22}$$

And an example where the average is unknown is for nowhere continuous f defined on Liouville Numbers [12].

How do we define a satisfying average for this case?

3.13 Questions on $\text{avg}_f(A)$

1. Has a unique, satisfying average for the largest class of measurable A been found? Does it match or extend $\text{avg}_f(A)$?

2. If neither is true, can we extend the definition of *structures* to uncountable sets and define a choice function which picks a group of equivalent structure such that it give a unique, satisfying average?

3. If this is also impossible, could we use non-standard measure theory ([4]) and Theorem 7 of "Integration with Filters" [13]) to create a choice function similar to eq:20, such that it reduces $\text{avg}_f(A)$ to one sub-function and

extends it by giving a unique, satisfying value when A is uncountable and non-fractal with no gauge function.