Average On Countably Infinite Sets

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0.1 Motivation

Suppose $f : A \to \mathbb{R}$, where A is a countably infinite set.

We can enumerate A as $\{a_n\}_{n=1}^{\infty}$ and set t as a natural number between one and infinity; however the average

$$\lim_{t\to\infty} \frac{f(a_1) + f(a_2) + \cdots + f(a_t)}{t}$$

can vary depending on the enumeration.

Therefore, I need to find a way to determine "structures" of A which give the same "intuitive average".

By "structures" I mean that if $F_1, F_2, \cdots$ are an infinite sequence of finite sets (denoted $\{F_n\}_{n=1}^{\infty}$) such that $F_1 \subseteq F_2 \subseteq \cdots$ and $\bigcup\limits_{n=1}^{\infty} F_n = A$ then F_t (for $t \in \mathbb{N}$ between one and infinity) is a structure of A.

By "average" I mean that we take the average of elements in F_t as t grows larger. In other words for the equation below:

$$\frac{1}{|F_t|} \sum_{x\in F_t} f(x)$$

substitute larger values of t and as it approaches infinity, see if it converges to an number. (I'm not sure how to describe this in pure mathematics).

By "intuitive", I wish for F_t to have the most "even distribution" possible.

For example, if $A = \mathbb{Q} \cap [0, 1]$, F_t could be $\left\{\frac{j}{k!} : j, k \in \mathbb{N}, j \le k! \le t\right\}$ since

$$\bigcup_{n=1}^{\infty} \left\{\frac{j}{k!} : j, k \in \mathbb{N}, j \le k! \le n\right\} = \mathbb{Q} \cap [0, 1]$$

and for every $t \in \mathbb{N}$ if the elements in F_t are arranged from least to greatest, the difference between pairs of consecutive elements must be close to equal as possible. For example, for $\left\{\frac{j}{k!} : j, k \in \mathbb{N}, j \leq k! \leq t\right\}$ note for any $k \in \mathbb{N}$, if $k = r$ is the largest positive integer where $k! \leq t$ and the resulting elements (arranged from least to greatest) are

$$F_n = \left\{\frac{0}{r!}, \frac{1}{r!}, \frac{2}{r!} \cdots, 1\right\}$$

The differences of the consecutive elements

$$\left\{\frac{1-0}{r!}, \frac{2-1}{r!}, \ldots, \frac{r! - (r!-1)}{r!}\right\}$$

$$\left\{\frac{1}{r!}, \frac{1}{r!}, \ldots, \frac{1}{r!}\right\}$$

end up being the same.

For most cases of A the difference between consecutive elements of F_t (for every $t \in \mathbb{N}$) can never be the same. But as long as we can cover all of A with $\bigcup_{n=1}^{\infty} F_n$ we want an F_t where (for elements arranged from least to greatest) the difference of pairs of consecutive elements is as close to equal as possible.

Hence, I need a rigorous way of determining a unique $\{F_n\}$ since there are several $\{F_n\}$ where $\bigcup\limits_{n=1}^{\infty} F_n = A$ but give different averages.

0.2 Attempt:

For every t in F_t, organize the elements from least to greatest and take the set of the difference of pairs of consecutive elements (we will call this ΔF_t).

For example, suppose $t = 10$, $F_{10} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\right\}$

Organize the elements from least to greatest:

$$\left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$$

Then take the difference of the consecutive elements:

$$\left\{\frac{1}{3} - 0, \frac{1}{2} - \frac{1}{3}, \frac{2}{3} - \frac{1}{2}, 1 - \frac{2}{3}\right\}$$

$$\Delta F_{10} = \left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}\right\}$$

Now suppose $\max^n$ represents the n-th largest element. For example if $\Delta F_{10} = \left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}\right\}$:

$$\max^1 (\Delta F_{10}) = \frac{1}{3}$$

Since $\frac{1}{3}$ is the largest element in $\left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}\right\}$. Since $\frac{1}{3}$ is also the second largest element we also have

$$\max^2 (\Delta F_{10}) = \frac{1}{3}$$

However, $\max^3 (\Delta F_{10}) = \frac{1}{6}$ and $\max^4 (\Delta F_n) = \frac{1}{6}$, since $\frac{1}{6}$ is both the third and fourth largest element.

Therefore for countably infinite A, to find a unique structure (or structures that give the same average), I need a $\{F_n\}$ which gives a value of the following (as t grows larger and see if it converges):

$$d(\{F_n\}, A) = \frac{1}{|F_t| - 1} \sum_{s=1}^{|F_t|-1} \sum_{t=1}^{s} \max^t (\Delta F_t)$$

that has the infimum of (as t grows larger)

$$\left| \frac{\sup (F_t) - \inf (F_t)}{2} - d(\{F_n\}, A) \right|$$

Making our average (as t grows larger):

$$\frac{1}{|F_t|} \sum_{x \in F_t} f(x)$$

0.3 Question

1. How do we rigorously express the average from $\{F_n\}$ and $d(\{F_n\}, A)$?

2. Does the average listed in the attempt give the average in the motivation?

3. How do we find a formula for calculating d in terms of A and $\{F_n\}$?

4. Depending on A, how do we find the F_n that gives the infimum of

$$\left| \frac{\sup (F_t) - \inf (F_t)}{2} - d(\{F_n\}, A) \right|$$

5. If none of assumptions are true, how should we approach my motivation instead?