Choosing $\{F_n\}$ For General A

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1 Motivation

Let f be a measurable function and let $A \subset \mathbb{R}^n$ be a measurable set.

Now let $\mathcal{M}$ be a collection of measurable sets and let $\mathcal{M}^+$ be the collection of all measurable sets with positive Lebesgue Measure. If λ is the Lebesgue Measure, the average of f is already defined for every $A \in \mathcal{M}^+$ with the formula.

$$m_f(A) = \frac{1}{\lambda(A)} \int_A f d\lambda$$

However, I wish to extend $m_f : \mathcal{M}^+ \to \mathbb{R}$ to the function $\dot{m}_f : \mathcal{M} \to \mathbb{R}$ which gives sensible results for sets with zero Lebesgue Measure that lie between the infimum and supremum of f.

There are several ways to do this but I wish to briefly focus on countable A with infinite points. What would an intuitive average look like?

2 Defining Density

Suppose A is an arbitrary, countable set with infinite points and suppose d is the Hausdorff Dimension with H^d being the Hausdorff Measure

Moreover, suppose $F_1, F_2, \cdots$ are a infinite sequence of sets denoted $\{F_n\}$, where $H^d(F_1), H^d(F_2), \cdots$ is positive and finite, $F_1 \subseteq F_2 \subseteq \cdots$, and $\bigcup\limits_{n=1}^{\infty} F_n = A$.

If S is a subset of A, the density of S is defined as

$$D(S, \{F_n\}) = \lim_{\omega \to \infty} \frac{H^d(S \cap F_\omega)}{H^d(F_\omega)}$$

We can use D to find the average of f.

Unfortunately, there are multiple $\{F_n\}$ we can define which prevents us from getting a unique average.

While I believe there is a way we can find one, I'm not sure it will give a unique $\{F_n\}$

3 Question

If $\bigcup_{n=1}^{\infty} F_n = A$, $1 \leq \omega \leq \infty$, and the average from the Hausdorff Measure is

$$\overline{F_\omega} = \frac{1}{H^d(A)} \int_A f(x) H^d$$

And if F_ω^2 squares all elements in F_ω, and the standard deviation of F_ω is

$$\sigma\{F_\omega\} = \sqrt{\overline{F_\omega^2} - \overline{F_\omega}^2}$$

Does there exist a group of F_ω with a $H^d(F_\omega)$ is less than ω, and close to ω as possible, such that their standard deviation has the smallest value for the most $\omega \leq t$ as $t \to \infty$; and their density of S or $d(S, \{F_n\})$ is the same?

And if find a unique $D(S, \{F_n\})$ for this case is impossible, how do we manipulate the above case to get a unique D.

For example if

$$A = \left\{ \frac{\sqrt{m}}{\sqrt{n}} : m, n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\ln(n)} : n \in \mathbb{N} \right\}$$

If possible, which F_ω would give the smallest standard deviation with a cardinality less than ω for the most $\omega \leq t$ as $t \to \infty$ such that $\bigcup_{n=1}^{\infty} F_n = A$? Would they give the same $d(S, \{F_n\})$?

If not possible, how can we find a unique $d(S, \{F_n\})$ using standard deviation.

4 Example:

If $A = \mathbb{Q} \cap [0, 1]$, an F_ω that gives the smallest standard deviation with a cardinality less than ω and closest to ω as $\omega \to \infty$ is

$$F_\omega = \left\{ \frac{r}{s!} : r, s \in \mathbb{Z}, r \leq s! \leq \left\lceil e \exp\left(\mathrm{W}\left(\frac{1}{e} \log\left(\frac{\omega}{\sqrt{2\pi}} \right) \right) + 1 \right) - \frac{1}{2} \right\rceil \right\}$$

Note if the cardinality of F_ω is less than ω as $\omega \to \infty$, $\bigcup\limits_{n=1}^{\infty} F_n = \mathbb{Q} \cap [0,1]$ and all F_ω have a standard deviation of zero.

I believe this is the only F_ω with the smallest standard deviation for every ω.