Choosing $\{F_n\}$ For Countable A

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May 2020

1 Motivation

Let f be a measurable function and let $A \subset \mathbb{R}^n$ be a measurable set.

Now let $\mathcal{M}$ be a collection of measurable sets and let $\mathcal{M}^+$ be the collection of all measurable sets with positive Lebesgue Measure. If λ is the Lebesgue Measure, the average of f is already defined for every $A \in \mathcal{M}^+$ with the formula.

$$m_f(A) = \frac{1}{\lambda(A)} \int_A f d\lambda$$

However, I wish to extend $m_f : \mathcal{M}^+ \to \mathbb{R}$ to the function $\dot{m}_f : \mathcal{M} \to \mathbb{R}$ which gives sensible results for sets with zero Lebesgue Measure that lie between the infimum and supremum of f.

There are several ways to do this but I wish to briefly focus on countable A with infinite points. What would an intuitive average look like?

2 Defining Density

Suppose A is an arbitrary, countable set with infinite points and suppose d is the Hausdorff Dimension with H^d being the Hausdorff Measure

Moreover, suppose $F_1, F_2, \cdots$ are a infinite sequence of sets denoted $\{F_n\}$, where $H^d(F_1), H^d(F_2), \cdots$ is positive and finite, $F_1 \subseteq F_2 \subseteq \cdots$, and $\bigcup_{n=1}^{\infty} F_n = A$.

If S is a subset of A, the density of S is defined as

$$D(S, \{F_n\}) = \lim_{\omega \to \infty} \frac{H^d(S \cap F_\omega)}{H^d(F_\omega)}$$

We can use D to find the average of f.

Unfortunately, there are multiple $\{F_n\}$ we can define which prevents us from getting a unique average.

While I believe there is a way we can find one, I'm not sure it will give a unique $\{F_n\}$

3 Question

Suppose we have the following:

- $\bigcup_{n=1}^{\infty} F_n = A$
- $1 \leq \omega \leq \infty$
- $C\left(\min\left\{F_\omega\right\}\right) = \left\{x : x, y \in F_\omega, x, y \neq \min\left\{F_\omega\right\}, x - \min\left\{F_\omega\right\} = \min\left\{y - \min\left\{F_\omega\right\}\right\}\right\}$
- C^n is the n-th iteration of C. For example $C^3\left(\min\left\{F_\omega\right\}\right)$ is $C(C(C(\min\left\{F_\omega\right\})))$
- $P = \left\{\bigcup_{n=1}^{H^d(F_\omega)-1} C^{n+1}\left(\min\left\{F_\omega\right\}\right) - C^n\left(\min\left\{F_\omega\right\}\right)\right\}$
- $T = \left\{\bigcup_{n=1}^{H^d(P)-1} C^{n+1}\left(\min\left\{P\right\}\right) - C^n\left(\min\left\{P\right\}\right)\right\}$
- $\overline{T} = \frac{1}{|T|} \sum_{x \in T} x$

And standard Deviation of T is

$$\sigma\left\{T\right\} = \sqrt{\frac{1}{|T| - 1} \sum_{x \in T} \left(x - \overline{T}\right)^2}$$

Does there exist a group of F_ω with a cardinality less than ω which has the smallest difference from ω, such that the standard deviation of T has the smallest value as $\omega \to \infty$; and their density of S or $d\left(S, \{F_n\}\right)$ for all the remaining F_ω?

If this doesn't make sense how can we manipulate this to get a unique $d\left(S, \{F_n\}\right)$

If not how do we manipulate this so we get a unique $d\left(S, \{F_n\}\right)$

For example if

$$A = \left\{\frac{\sqrt{m}}{\sqrt{n}} : m, n \in \mathbb{N}\right\} \cup \left\{\frac{1}{\ln(n)} : n \in \mathbb{N}\right\}$$

Which F_ω would give the smallest standard deviation of T with a cardinality less than ω as $\omega \to \infty$ such that $\bigcup_{n=1}^{\infty} F_n = A$? Would they give the same $d(S, \{F_n\})$?

3.1 Example

For example, if $A = \mathbb{Q} \cap [0,1]$, an F_ω that gives the smallest standard deviation with a cardinality greater than ω and closest to ω as $\omega \to \infty$ is

$$F_\omega = \left\{ \frac{r}{s!} : r, s \in \mathbb{N}, r \leq s! \leq \left\lceil e \exp\left(\mathrm{W}\left(\frac{1}{e} \log\left(\frac{\omega}{\sqrt{2\pi}} \right) \right) + 1 \right) - \frac{1}{2} \right\rceil \right\}$$

Note if the cardinality of F_ω is less than ω as $\omega \to \infty$, $\bigcup_{n=1}^{\infty} F_n = \mathbb{Q} \cap [0,1]$ and all F_ω have a standard deviation for every ω.

I believe this is the only F_ω with the smallest standard deviation for every ω.