Thought Process For Choosing $\{F_n\}$ when A is countable

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1 Motivation

Let f be a measurable function and let $A \subset \mathbb{R}^n$ be a measurable set.

Now let $\mathcal{M}$ be a collection of measurable sets and let $\mathcal{M}^+$ be the collection of all measurable sets with positive Lebesgue Measure. If λ is the Lebesgue Measure, the average of f is already defined for every $A \in \mathcal{M}^+$ with the formula.

$$m_f(A) = \frac{1}{\lambda(A)} \int_A f d\lambda$$

However, I wish to extend $m_f : \mathcal{M}^+ \to \mathbb{R}$ to the function $\dot{m}_f : \mathcal{M} \to \mathbb{R}$ which gives sensible results for sets with zero Lebesgue Measure that lie between the infimum and supremum of f.

There are several ways to do this but I wish to briefly focus on countable A with infinite points. What would an intuitive average look like?

2 Defining Density

Suppose A is an arbitrary, countable set with infinite points.

Moreover, suppose $F_1, F_2, \cdots$ is a infinite sequence of finite sets denoted $\{F_n\}$ such that $F_1 \subseteq F_2 \subseteq \cdots$ and $\bigcup\limits_{n=1}^{\infty} F_n = A$.

If S is a subset of A, the density of S is defined as

$$d(S, \{F_n\}) = \lim_{\omega \to \infty} \frac{|S \cap F_\omega|}{|F_\omega|}$$

We can use d to find the average of f when A is countable.

Unfortunately, there are multiple $\{F_n\}$ we can define which prevents us from getting a unique average.

While I believe there is a way we can find one, I'm not sure it will give a unique $\{F_n\}$

3 Question

If $\bigcup\limits_{n=1}^{\infty} F_n = A$, $1 \leq \omega \leq \infty$, the arithmetic mean of F_ω is

$$\overline{F}_\omega = \frac{1}{|F_\omega|} \sum_{x \in F_\omega} x$$

And standard Deviation of F_ω is

$$\sigma\left\{F_\omega\right\} = \sqrt{\frac{1}{|F_\omega| - 1} \sum_{x \in F_\omega} \left(x - \overline{F}_\omega\right)^2}$$

Does there exist a group of F_ω with a cardinality less than ω, and close to ω as possible, such that their standard deviation has the smallest value for the most $\omega \leq t$ as $t \to \infty$; and their density of S or $d\left(S, \{F_n\}\right)$ is the same?

For example if

$$A = \left\{\frac{\sqrt{m}}{\sqrt{n}} : m, n \in \mathbb{N}\right\} \cup \left\{\frac{1}{\ln(n)} : n \in \mathbb{N}\right\}$$

Which F_ω would give the smallest standard deviation with a cardinality less than ω for the most $\omega \leq t$ as $t \to \infty$ such that $\bigcup\limits_{n=1}^{\infty} F_n = A$? Would they give the same $d(S, \{F_n\})$?

4 Example:

If $A = \mathbb{Q} \cap [0,1]$, an F_ω that gives the smallest standard deviation with a cardinality greater than ω and closest to ω as $\omega \to \infty$ is

$$F_\omega = \left\{\frac{r}{s!} : r, s \in \mathbb{Z}, r \leq s! \leq \left\lceil e \exp\left(\mathrm{W}\left(\frac{1}{e}\log\left(\frac{\omega}{\sqrt{2\pi}}\right)\right) + 1\right) - \frac{1}{2}\right\rceil\right\}$$

Note if the cardinality of F_ω is less than ω as $\omega \to \infty$, $\bigcup\limits_{n=1}^{\infty} F_n = \mathbb{Q} \cap [0,1]$ and all F_ω have a standard deviation of zero.

I believe this is the only F_ω with the smallest standard deviation of zero.