

Generalizing the Arithmetic Mean

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May 2020

0.1 Suggestion

Don't treat this as a research paper. Lightly skim.

0.2 Theories One Should Know

- Descriptive Set Theory
- Measure Theory
- Box-Counting Dimensions
- Folner Sequences

1 Purpose

- *I wish to create a measure that calculates arithmetic mean for functions where it's well-defined but calculates a new arithmetic mean for functions where it's not.*

I wish to find a measure (possibly new) that calculates a unique arithmetic mean for the largest class of functions and coincides with the arithmetic mean of other functions.

2 Problems With Other Definitions

2.1 Lebesgue Measure

Suppose $f : A \rightarrow B$ is an arbitrary measurable function, where A is a non-fixed arbitrary subset of the real numbers.

- The Lebesgue Measure of A or $\lambda(A)$ gives the average $\frac{1}{\lambda(A)} \int_A f(x) d\lambda$ and is undefined when $\lambda(A) = 0$.

2.2 Counting Measure

- The counting measure, μ , of the domain, A , is denoted $\mu(A)$ and $|A|$ is the cardinality of A such that

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

and the average $\frac{1}{\mu(A)} \sum_{x \in A} f(x)$ is undefined when $\mu(A) = \infty$.

- Hence when generalizing the arithmetic mean we consider functions whose domain has infinite points.

2.3 Hausdorff Measure

The Hausdorff Measure seems like a proper solution but there lies an issue: Suppose the domain has zero Lebesgue Measure and infinite elements.

- If we define the average on A using the Hausdorff Measure, we must rely on its dimension to get a defined mean.
- If the dimension is zero, we get the counting measure (this gives a defined mean when A is finite).
- If the dimension is one, we get the Lebesgue Measure (this gives a defined mean when A has a measure greater than zero).
- When A has zero Lebesgue Measure with infinite points there exists a dimension $d \in [0, 1]$ where $s < d$ give a measure of ∞ and $s > d$ give a measure of 0.
- If $d = s$ the Hausdorff Measure would be any number between zero and infinity and is therefore not unique.

2.4 Conditional Expectation

- When applying the conditional expectation for finding the average of f (in the measure-theoretic sense) one must find a distribution of A .
- If A has zero Lebesgue Measure and infinite elements, I assume there is multiple distributions to choose for defining the mean. Therefore we must apply the axiom of choice to get a unique mean.

I assume we are unable to come up with a choice function that gives a unique distribution for every A and in turn gives a unique measure for conditional expectation.

2.5 Properties Our Measure Should Have

If S is an arbitrary subset of A and A is an arbitrary subset of \mathbb{R} , we should define a probability measure $\mathcal{P}(S, A)$ which yields an arithmetic mean that coincides with other arithmetic means. The measure should have the following properties.

- When $S = A$, $\mathcal{P}(S, A) = \mathcal{P}(A, A) = 1$
- When $\lambda(A) \geq 0$, $\mathcal{P}(S, A) = \lambda(S)/\lambda(A)$
- When S is countable and A is uncountable, $\mathcal{P}(S, A) = 0$ (regardless if $\lambda(A) \geq 0$).
- When $\lambda(S) = \lambda(A)$, with S being nowhere dense and A being dense in an interval, then $\mathcal{P}(S, A) = 0$

3 First Attempt In Generalizing Arithmetic Mean

3.1 Defining Measure \mathcal{P} (Try and Read The Whole Section)

If $S \subseteq A$:

- ℓ is the length of an interval
- $(I_j)_{j=1}^m$ and $(J_k)_{k=1}^n$, for $m, n \in \mathbb{N}$, are a sequence of open intervals where $\ell(I_1) = \dots = \ell(I_m) = g \in \mathbb{R}^+$, $\ell(J_1) = \dots = \ell(J_n) = g \in \mathbb{R}^+$ and the infimum of equations below are taken over all possible I_j and J_k

$$\mathcal{M}(g, S) = \begin{cases} g \cdot \inf \left\{ m \in \mathbb{N} : S' \subseteq S, S' \text{ is countable, } (S \setminus S') \subseteq \bigcup_{k=1}^m I_m \right\} & A \text{ is uncountable} \\ g \cdot \inf \left\{ m \in \mathbb{N} : S \subseteq \bigcup_{k=1}^m I_m \right\} & A \text{ is countable} \end{cases} \quad (1)$$

and

$$\mathcal{N}(S, A) = \lim_{g \rightarrow 0} \frac{\mathcal{M}(g, S)}{\mathcal{M}(g, A)} \quad (2)$$

then we add $\mathcal{O}(g, S)$ so if A is a interval with finite length, $\mathcal{O}(g, S)$ gives the Lebesgue Measure of S over the Lebesgue Measure of A .

$$\mathcal{O}(g, S) = \begin{cases} g \cdot \inf \left\{ n \in \mathbb{N} : S_j \subseteq S, \mathcal{N}(S_j, A) = 0, \bigcup_{j=1}^{\infty} S_j = S', (S \setminus S') \subseteq \bigcup_{k=1}^n J_k \right\} & A \text{ is uncountable} \\ g \cdot \inf \left\{ n \in \mathbb{N} : S \subseteq \bigcup_{k=1}^n J_n \right\} & A \text{ is countable} \end{cases} \quad (3)$$

therefore the outer measure is

$$\mathcal{P}^*(S, A) = \lim_{g \rightarrow 0} \frac{\mathcal{O}(g, S)}{\mathcal{O}(g, A)} \quad (4)$$

And the inner measure is $\mathcal{P}_*(S, A) = 1 - \mathcal{P}^*(A \setminus S, A)$, meaning measure $\mathcal{P}(S, A)$ exists when $\mathcal{P}^*(S, A) = \mathcal{P}_*(S, A)$.

3.2 Defining \mathcal{P} -Average

When $\mathcal{P}(S, A)$ is defined, the \mathcal{P} -average of $f : A \rightarrow B$ or \bar{f} is defined as follows:

Split $[a, b] = [\min(A), \max(A)]$ into sub-intervals using partitions x_i

$$a = x_0 \leq \dots \leq x_i \leq \dots \leq x_r = b$$

Here, if $1 \leq i \leq r$, then $[x_{i-1}, x_i]$ are sub-intervals of $[a, b]$.

For every i , choose a $v_i \in A \cap [x_{i-1}, x_i]$ and define $A_i = A \cap [x_{i-1}, x_i]$. Then define set P , such that $i \in P \subseteq \{1, \dots, r \in \mathbb{N}\}$ when $A \cap [x_{i-1}, x_i] \neq \emptyset$, giving

$$\bar{f}(x) = \lim_{r \rightarrow \infty} \sum_{i \in P} f(v_i) \times \mathcal{P}(A_i, A) \quad (5)$$

(This looks tedious but one can use shortcuts to simplify the sum. If I write it out post it will be too long.)

3.3 Examples Not Well-Defined By Current Definitions of Mean But Defined By My Definition (of Mean)

3.4 First Example

Before defining function f , I will define sub-domains:

Suppose we set $G_0 = [0, 1]$. We define $(G_0 + c)/d$ where if $c, d \in \mathbb{R}$, add every element in $x \in G_0$ by c and divide by d . Therefore, if we define $n \in \mathbb{Z}^+$, $c \in \{0, 2, 4\}$ and $d = 5$ then we extend G_0 to the recursion sequence G_{n+1} .

$$G_{n+1} = \frac{G_n}{5} \cup \frac{G_n + 2}{5} \cup \frac{G_n + 4}{5} \quad (6)$$

The first sub-domain, defined \mathcal{G} , is

$$\mathcal{G} = \bigcap_{n=0}^{\infty} G_n$$

The second sub-domain shall be \mathcal{C} or the Cantor Set and the third sub-domain is $\mathbb{Q} \cap [0, 1]$.

Therefore, we define $f : (\mathcal{C} \cup \mathcal{G} \cup \mathbb{Q}) \cap [0, 1] \rightarrow \{1, 2, 3\}$:

$$f_2(x) = \begin{cases} 3 & x \in \mathcal{G} \setminus \mathbb{Q} \\ 2 & x \in \mathcal{C} \setminus (\mathcal{G} \cup \mathbb{Q}) \\ 1 & x \in \mathbb{Q} \cap [0, 1] \end{cases}$$

3.5 Second Example

Consider function f where for its first sub-domain S_1 ,

$$S_1 = \left\{ \frac{1}{\sqrt{2s^2}} : s \in \mathbb{N} \right\}$$

And its second domain S_2

$$S_2 = \left\{ \frac{1}{2^s} + \frac{1}{2^t} : s, t \in \mathbb{N} \right\}$$

Therefore, $f_4 : S_1 \cup S_2 \rightarrow \{0, 1\}$

$$f_4(x) = \begin{cases} 1 & x \in S_1 \\ 0 & x \in S_2 \end{cases}$$

3.6 Example Where \mathcal{P} Is Undefined

Consider the following:

If $f : \left\{ \frac{1}{s} : s \in \mathbb{N} \right\} \rightarrow \{0, 1\}$ and

$$f(x) = \begin{cases} 1 & x \in \left\{ \frac{1}{2^{s+1}} : s \in \mathbb{N} \right\} \\ 0 & x \in \left\{ \frac{1}{2^s} : s \in \mathbb{N} \right\} \end{cases}$$

Suppose $S_1 = \left\{ \frac{1}{2^{s+1}} : s \in \mathbb{N} \right\}$ and $S_2 \in \left\{ \frac{1}{2^s} : s \in \mathbb{N} \right\}$

Using \mathcal{P}^* , I approximated (and must prove using box-counting dimensions) $\mathcal{P}^*(S_1, S_1 \cup S_2) \geq 1/\sqrt{2}$ and $\mathcal{P}^*(S_2, S_1 \cup S_2) \geq 1/\sqrt{2}$. Hence, $\mathcal{P}_*(S_1, S_1 \cup S_2) = 1 - \mathcal{P}^*(S_2, S_1 \cup S_2) \leq 1 - 1/\sqrt{2}$ but $1/\sqrt{2} \neq 1 - 1/\sqrt{2}$. Hence with this case my measure is undefined.

Instead we need a measure that gives the same value as $\mathcal{P}(S, A)$ when \mathcal{P} is defined but gives different value when $\mathcal{P}(S, A)$ is undefined.

3.7 Defining Density

Suppose $A \subseteq \mathbb{R}$ and $S \subseteq A$.

And suppose F_1, \dots, F_ω are a sequence of sets denoted $\{F_n\}$ where $\mathcal{P}(F_1, A), \dots, \mathcal{P}(F_\omega, A)$ are defined, $F_1 \subseteq F_2 \subseteq \dots \subseteq F_\omega$ and $\lim_{\omega \rightarrow \infty} \bigcup_{n=1}^{\omega} F_n = A$

We write density of S as

$$d(S, \{F_n\}) = \lim_{\omega \rightarrow \infty} \mathcal{P}(S \cap F_\omega, F_\omega)$$

The problem is there are several ways to write $\{F_n\}$. For $A = \mathbb{Q} \cap [0, 1]$, $\{F_n\}$ can be written as:

$$\{F_n\} = \left\{ \frac{m}{n} : m, n \in \mathbb{N}, m \leq n \leq \omega \right\}$$

Since $\left\{ \frac{m}{n} : m, n \in \mathbb{N} \right\} = \mathbb{Q}$

or

$$\{F_n\} = \left\{ \frac{m}{n^2} : m, n \in \mathbb{N}, m \leq n^2 \leq \omega \right\}$$

Since $\left\{ \frac{m}{n^2} : m, n \in \mathbb{N} \right\} = \mathbb{Q}$

Moreover, when A is countable and dense in an interval and S is countable and almost nowhere dense, one can come up $\{F_n\}$ where the density of S is non-zero. (This is non-intuitive.)

For instance, consider $A = \left(\mathbb{Q} \cup \left\{ \frac{1}{\ln(n)} : n \in \mathbb{N} \right\} \right) \cap [0, 1]$ and $S = \left\{ \frac{1}{\ln n} : n \in \mathbb{N} \right\} \cap [0, 1]$. One can come up with $\{F_n\} = \left\{ \frac{m}{n} : m, n \in \mathbb{N}, n \neq 0, m \leq n \leq \omega \right\} \cup \left\{ \frac{1}{\ln(n)} : n \in \mathbb{N}, 0 < \ln(n) \leq \omega \right\}$ which makes $d(S, \{F_n\})$ (I assume this would be simple to prove but would like to check using mathematical proof)

Therefore, we must define a new density, where if A is dense in an interval, and S is almost nowhere dense, then S shall always have zero density.

3.8 Defining New Density

If $\psi \in A'$ where A' is the derived set of A and $|\cdot|$ is the absolute value, we define the set of \mathcal{P} measurable-sequences as

$$B(\psi, \epsilon, \omega, \{F_n\}) = \left\{ \alpha : \alpha \in \bigcup_{n=1}^{\omega} F_n, |\alpha - \psi| \leq \epsilon \right\}$$

which is in the interval $[\psi - \epsilon, \psi + \epsilon]$

To get a unique $\{F_n\}$, we must define a choice function that can be applied to arbitrary A and coincide with $\mathcal{P}(S, A)$ when $\mathcal{P}(S, A)$ is defined.

I'm don't know how this can be done but here is my attempt: (Skip If You Don't Understand)

3.8.1 Attempt

Note the density without a choice function is simply:

$$d(S, \{F_n\}, \psi) = \lim_{\epsilon \rightarrow 0} \lim_{\omega \rightarrow \infty} \mathcal{P}(S \cap B(\psi, \epsilon, \omega, \{F_n\}), B(\psi, \epsilon, \omega, \{F_n\}))$$

[I assume] we can define a choice function for any arbitrary A by doing the following:

If σ is the standard deviation of a distribution such that if X is a non-fixed arbitrary set and \mathcal{P} -measurable, and \bar{X} is the \mathcal{P} -average of X (Section 3.2) with X^2 squaring all elements in X then

$$\sigma \{X\} = \sqrt{\bar{X^2} - \bar{X}^2}$$

Therefore, if $\mathcal{F}(A)$ represents all sequences of sets with the same properties as $\{F_n\}$ but written in different forms, $\{H_n\}, \{G_n\} \in \mathcal{F}(A)$ and:

$$B(\psi, \epsilon, \omega_1, \{G_n\}) = \left\{ \alpha : \alpha \in \bigcup_{n=1}^{\omega_1} G_n, |\alpha - \psi| \leq \epsilon \right\}$$

$$B(\psi, \epsilon, \omega_2, \{H_n\}) = \left\{ \alpha : \alpha \in \bigcup_{n=1}^{\omega_2} H_n, |\alpha - \psi| \leq \epsilon \right\}$$

$$\begin{aligned} \beta = & \left\{ \{G_n\} : \forall (\{H_n\}) \forall (\omega_2) \exists (\omega_1) \left(\lim_{\epsilon \rightarrow 0} \lim_{\omega_2 \rightarrow \infty} \frac{\sigma \{B(\psi, \epsilon, \omega_1, \{G_n\})\}}{\sigma \{B(\psi, \epsilon, \omega_2, \{H_n\})\}} \leq 1 \right) \right. \\ & \forall (H_n) \forall (\omega_2) \exists (\omega_1) \left(\frac{\lim_{\epsilon \rightarrow 0} \lim_{\omega_2 \rightarrow \infty} \mathcal{P} \left(B(\psi, \epsilon, \omega_1, \{G_n\}), B(\psi, \epsilon, \omega_1, \{G_n\}) \cup B(\psi, \epsilon, \omega_2, \{H_n\}) \right)}{\mathcal{P} \left(B(\psi, \epsilon, \omega_2, \{H_n\}), B(\psi, \epsilon, \omega_1, \{G_n\}) \cup B(\psi, \epsilon, \omega_2, \{H_n\}) \right)} = \right. \\ & \left. \left. \left. \inf \left\{ |1 - s| : s = \lim_{\epsilon \rightarrow 0} \lim_{\omega_2 \rightarrow \infty} \frac{\mathcal{P} \left(B(\psi, \epsilon, \omega_3, \{G_n\}), B(\psi, \epsilon, \omega_3, \{G_n\}) \cup B(\psi, \epsilon, \omega_2, \{H_n\}) \right)}{\mathcal{P} \left(B(\psi, \epsilon, \omega_2, \{H_n\}), B(\psi, \epsilon, \omega_3, \{G_n\}) \cup B(\psi, \epsilon, \omega_2, \{H_n\}) \right)}, s < 1 \right\} \right\} \right\} \end{aligned}$$

What this means is we want to find all $\{G_n\}$ where for all $\{H_n\}$ and ω_2 there exists an ω_1 such that

$$\lim_{\epsilon \rightarrow 0} \lim_{\omega_2 \rightarrow \infty} \frac{\sigma \{B(\psi, \epsilon, \omega_1, \{G_n\})\}}{\sigma \{B(\psi, \epsilon, \omega_2, \{H_n\})\}} \leq 1$$

and for all $\{H_n\}$ and ω_2 there exists an ω_1 such that if

$$s = \lim_{\epsilon \rightarrow 0} \lim_{\omega_2 \rightarrow \infty} \frac{\mathcal{P} \left(B(\psi, \epsilon, \omega_1, \{G_n\}), B(\psi, \epsilon, \omega_1, \{G_n\}) \cup B(\psi, \epsilon, \omega_2, \{H_n\}) \right)}{\mathcal{P} \left(B(\psi, \epsilon, \omega_2, \{H_n\}), B(\psi, \epsilon, \omega_1, \{G_n\}) \cup B(\psi, \epsilon, \omega_2, \{H_n\}) \right)}$$

and $s < 1$ then s should be as close to 1 as possible.

Hence, with $\{F_n\} \in \beta$, we get the density:

$$d(S, \{F_n\}, \psi) = \lim_{\epsilon \rightarrow 0} \lim_{\omega \rightarrow \infty} \mathcal{P}(S \cap B(\psi, \epsilon, \omega, \{F_n\}), B(\psi, \epsilon, \omega, \{F_n\}))$$

I assume this would give us a unique value.

3.8.2 Attempt Finished

However, even when $d(S, \{F_n\}, \psi)$ exists and gives a unique value, it only takes the density over $[\psi - \epsilon, \psi + \epsilon]$. Therefore, we broaden the density to all A by taking the weighted average of $d(S, \{F_n\}, \psi)$ using measure \mathcal{P} . In formal terms:

Divide $[0, 1]$ ($0 \leq d(S, \psi) \leq 1$) into partitions $0 = c_1 \leq \dots \leq c_i \leq \dots \leq c_u = 1$ where if $x_i \in [c_{i-1}, c_i]$ and $C_i = [c_{i-1}, c_i]$, our final upper density

$$\mathcal{D}^*(S, A, \{F_n\}) = \lim_{u \rightarrow \infty} \sum_{i=1}^u x_i \cdot \mathcal{P}^* (\{\psi : d(S, \{F_n\}, \psi) = C_i\}, A) \quad (7)$$

And our final lower density is

$$\mathcal{D}_*(S, A, \{F_n\}) = 1 - \lim_{u \rightarrow \infty} \sum_{c=1}^u x_i \cdot \mathcal{P}^* (\{\psi : d(A \setminus S, \{F_n\}, \psi) = C_i\}, A) \quad (8)$$

Therefore, our final density $\mathcal{D}(S, A, \{F_n\})$ exists when:

$$\mathcal{D}^*(S, A, \{F_n\}) = \mathcal{D}_*(S, A, \{F_n\})$$

3.9 Defining The Average

To define the average, first split $[a, b] = [\min(A), \max(A)]$ into sub-intervals using partitions x_i

$$a = x_0 \leq \dots \leq x_i \leq \dots \leq x_r = b$$

Here, if $1 \leq i \leq r$, then $[x_{i-1}, x_i]$ are sub-intervals of $[a, b]$.

For every i , choose a $v_i \in A \cap [x_{i-1}, x_i]$ and define $A_i = A \cap [x_{i-1}, x_i]$. Next we define set P , such that $i \in P \subseteq \{1, \dots, r \in \mathbb{N}\}$ when $A \cap [x_{i-1}, x_i] \neq \emptyset$. This gives

$$\lim_{r \rightarrow \infty} \sum_{i \in P} f(v_i) \times \mathcal{D}(A_i, A) \tag{9}$$

(This may look tedious but we can use shortcuts to simplify the sum. If I write it out the post will be too long.)

4 Examples

Here are section where one must apply the density and average:

4.1 Third Example

Suppose we want to define f_3 such that if \mathcal{C} is the Cantor Set, the first sub-domain of f_3 is:

$$S_n = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} \{3^{-n}(x+k) : x \in \mathcal{C}\}$$

If \mathcal{G} is the first sub-domain in second example (section 3.4), the second sub-domain of f_3 is:

$$T_n = \left(\bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \{5^{-n}(x+k) : x \in \mathcal{G}\} \right) \setminus S_n$$

Therefore $f_3 : S_n \cup T_n \rightarrow \{1, 2\}$

$$f_3(x) = \begin{cases} 2 & x \in S_n \\ 1 & x \in T_n \end{cases}$$

4.2 Fourth Example

Consider function $f_4(x)$ where $f_4 : \mathbb{Q} \cap [0, 1] \rightarrow \{0, 1\}$

$$f_4(x) = \begin{cases} 1 & x \in \left(\mathbb{Q} \setminus \left\{ \frac{s}{2t+1} : s, t \in \mathbb{N} \right\} \right) \cap [0, 1] \\ 0 & x \in \left\{ \frac{s}{2t+1} : s, t \in \mathbb{N} \right\} \cap [0, 1] \end{cases}$$