1.10 Finite, Infinite, Countable, and Uncountable Sets

Definition 1.10.1 Let A and B be two sets. We say that A and B have the same cardinality if there exists a bijection from A to B. In such a case, we say that A and B are equivalents and denote $A \sim B$.

We may define a relation, defining that A is related to B if $A \sim B$. We may show that such a relation is reflexive, symmetric, and transitive.

Observe that

- $A \sim A$ since $f : A \to A$ defined by $f(x) = x$ is a bijection.
- $A \sim B \Rightarrow B \sim A$, since if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection as well.
- $A \sim B$ and $B \sim C \Rightarrow A \sim C$, since if $f : A \to B$ and $g : B \to C$ are bijections we have that $h = g \circ f : A \to C$ is a bijection from A to C.

We recall that for each $n \in \mathbb{N}$ we earlier had denoted:

$$I_n = \{1, 2, \ldots, n\}.$$

Definition 1.10.2 Let A be a nonempty set.

1. We say that A is finite if $A \sim I_n$, for some $n \in \mathbb{N}$.
2. If A is not finite it is said to be infinite.
3. We say that A is countable if it is finite or if there exists a bijection from A to $\mathbb{N}$, that is, $A \sim \mathbb{N}$.
4. If A is not countable it is said to be uncountable.

As an example, consider $A = \mathbb{Z}$, the set of integers. We may show that $\mathbb{Z}$ is countable. Consider a bijection $f : \mathbb{N} \to \mathbb{Z}$ given by:

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ -(n-1)/2, & \text{if } n \text{ is odd,} \end{cases} \tag{1.24}$$

Clearly such a function is injective and surjective, so that

$$\mathbb{Z} = \{f(n) \mid n \in \mathbb{N}\} = \{1, 2, 3, 4, \ldots.\} \cup \{0, -1, -2, -3, \ldots\}.$$

Hence $\mathbb{Z}$ is countable.

Definition 1.10.3 (Sequence) All function whose domain is $\mathbb{N}$ is said to be a sequence. Thus $f : \mathbb{N} \to A$ is a sequence in A. We also denote $f(n) = x_n$ or the sequence simply by $\{x_n\}$.

Theorem 1.10.4 *Let A be countable. Assume that $E \subset A$ and that E is infinite. Under such hypotheses E is countable.*

Proof By hypothesis, A is countable and infinite. Hence A may be expressed by a sequence of distinct elements, since $A \sim \mathbb{N}$, that is, $A = \{x_n\}_{n \in \mathbb{N}}$. Let n_1 be the smallest natural such that $x_{n_1} \in E$. Reasoning inductively, having $n_1 < n_2 < \cdots < n_{k-1}$ define n_k as the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$. Define $f : \mathbb{N} \to E$ by

$$f(k) = x_{n_k}.$$

Being the elements of$\{x_n\}$ distinct, we have that f is injective. Let us show that f is also surjective. Let $x_j \in E$. Define k_0 as the greatest natural number such that $n_{k_0} < j$. Since $x_j \in E$ we obtain $n_{k_0+1} = j$, that is, $x_j \in \{x_{n_k}\}$, so that $E \subset \{x_{n_k}\}$. Since by definition $\{x_{n_k}\} \subset E$, we obtain $E = \{x_{n_k}\}$, so that f is surjective. The proof is complete.

Definition 1.10.5 Let A be a set of indices such that for each $\alpha \in A$ we associate an unique set denoted by E_α. The union of the sets E_α we shall denote by S so that

$$S = \cup_{\alpha \in A} E_\alpha.$$

Thus $x \in S \Leftrightarrow x \in E_\alpha$ for some $\alpha \in A$. If $A = \{1, \ldots, n\}$, we write

$$S = \cup_{i=1}^{n} E_i,$$

and if $A = \mathbb{N}$ we write

$$S = \cup_{n=1}^{\infty} E_n.$$

By analogy, the intersection between the sets E_α will be denoted by P, that is

$$P = \cap_{\alpha \in A} E_\alpha.$$

Thus

$$x \in P \Leftrightarrow x \in E_\alpha, \ \forall \alpha \in A.$$

If $A = \{1, \ldots, n\}$ we write

$$P = \cap_{i=1}^{n} E_i = E_1 \cap E_2 \cap \cdots \cap E_n.$$

If $A = \mathbb{N}$ we write,

$$P = \cap_{n=1}^{\infty} E_n.$$

Finally, if $A \cap B = \emptyset$ we say that A and B are disjoint.

Theorem 1.10.6 *Let $\{E_n\}$ be a sequence of countable sets. Under such hypotheses, $S = \cup_{n=1}^{\infty} E_n$ is countable.*

Proof We give just a sketch of the proof. Observe that for each $n \in \mathbb{N}$ E_n is countable, so that we may denote

$$E_n = \{x_{n_k}\}_{k \in \mathbb{N}}.$$

Hence,

$$E_1 = \{x_{11}, \ , x_{12}, \ x_{13}, \ \cdots\}$$

$$E_2 = \{x_{21}, \ , x_{22}, \ x_{23}, \ \cdots\}$$

$$E_3 = \{x_{31}, \ , x_{32}, \ x_{33}, \ \cdots\}$$

$$E_4 = \ \cdots \ \cdots \ \cdots \ \cdots \ \cdots$$

$$\cdots = \ \cdots \ \cdots \ \cdots \ \cdots \ \cdots$$

We may pass an arrow through x_{11} and define values for a function $f : \mathbb{N} \to S$ by setting $f(1) = x_{11}$. After that, we may pass a diagonal arrow from x_{21} to x_{12} and define $f(2) = x_{21}, \ f(3) = x_{12}$. We may pass a third arrow through x_{31}, x_{22}, and x_{13} and define $f(4) = x_{31}, \ f(5) = x_{22}, \ f(6) = x_{13}$. Proceeding in this fashion, we continue to pass diagonal arrows, associating a natural number through f, as an element of the table is touched by a concerned arrow. Observe that to each element $S = \cup_{n=1}^{\infty} E_n$ will be associated a natural number (Fig. 1.6).

If there exist repeated elements in the table above defined by S, we may infer that f will be a bijection between S and a subset T of $\mathbb{N}$. Hence, from the last theorem:

$$\mathbb{N} \sim T \sim S,$$

Thus

$$\mathbb{N} \sim S,$$

that is, $S = \cup_{n=1}^{\infty} E_n$ is countable.

Theorem 1.10.7 *Let A be a countable set. Then, $A \times A$ is a countable set.*

Proof The case in which A if finite is immediate. Thus assume A is infinite. Since A is countable we may denote $A = \{x_n\}$. Let $n \in \mathbb{N}$. Define $E_n = \{(x_n, x_m) \mid m \in \mathbb{N}\}$. Consider a bijection $f : E_n \to A$ defined by $f(x_n, x_m) = x_m$. Thus $E_n \sim A \sim \mathbb{N}$. Therefore, $E_n \sim \mathbb{N}$, that is E_n is countable, $\forall n \in \mathbb{N}$. Since

$$A \times A = \cup_{n=1}^{\infty} E_n$$

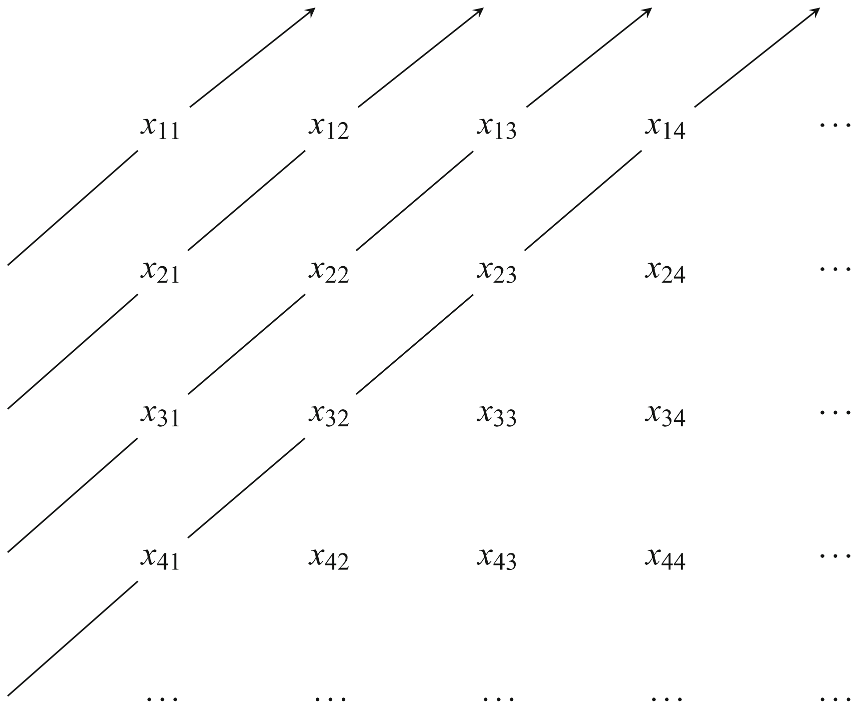

Fig. 1.6 Countability of a countable union of countable sets

from the last theorem we may conclude that $A \times A$ is countable.
The proof is complete.

Exercise 1.10.8 Show that if A and B are countable sets, then so is $A \times B$.

Exercise 1.10.9 Show that if A is countable and $n \in \mathbb{N}$, then $A^n = A \times A \times \cdots \times A$ (n times) is also countable, where we denote $A^2 = A \times A$ and $A^n = A^{n-1} \times A$.
Hint: Use induction.

Theorem 1.10.10 *The rational set $\mathbb{Q}$ is countable.*

Proof We have already proven that $\mathbb{Z}$ is countable and therefore from Theorem 1.10.7, $\mathbb{Z} \times \mathbb{Z}$ is countable. Consider the subset $A \subset \mathbb{Z} \times \mathbb{Z}$ defined by:

$$A = \{(m, n) \in \mathbb{Z} \times \mathbb{N} \mid g.c.d.(m, n) = 1\}.$$

Thus, since A is a subset of a countable set, it is also countable. Consider the function $f : A \to \mathbb{Q}$ defined by

$$f(m, n) = m/n.$$

Thus, f is a bijection from A to $\mathbb{Q}$. Therefore, $\mathbb{N} \sim A \sim \mathbb{Q}$, that is,

$$\mathbb{N} \sim \mathbb{Q}$$

so that $\mathbb{Q}$ is countable. The proof is complete.

Theorem 1.10.11 *The set* A *of all sequences whose elements are just* 0 *or* 1 *is uncountable.*

Proof Suppose, to obtain contradiction that A is countable. Hence, we may write $A = \{S_n\}_{n\in\mathbb{N}}$ where $S_n = \{s_{nk}\}_{k\in\mathbb{N}}$ and where $s_{nk} = 0$ or $s_{nk} = 1, \ \forall \ k, n \in \mathbb{N}$. Define the sequence $\tilde{S} = \{\tilde{s}_k\}$ by

$$\tilde{s}_k = \begin{cases} 1, & \text{if } s_{kk} = 0, \\ 0, & \text{if } s_{kk} = 1. \end{cases} \tag{1.25}$$

Thus $\tilde{S} \neq S_n, \forall n \in \mathbb{N}$. Hence, $\tilde{S} \notin A$. However, by its definition $\tilde{S} \in A$. We have got a contradiction. The proof is complete.

Corollary 1.10.12 $\mathbb{R}$ *is uncountable.*

Proof Consider the interval $[0, 0.2]$ and the decimal expansions of the form $0.\ x_1\ x_2 \cdots x_k \cdots$ where $x_k = 0$ or $x_k = 1, \ \forall k \in \mathbb{N}$. From the last theorem the collection of such numbers is uncountable. Therefore the interval $[0, 0.2]$ is uncountable so that $\mathbb{R} \supset [0, 0.2]$ is uncountable.

Exercise 1.10.13 Show that $\mathbb{R} \setminus \mathbb{Q}$, the set of irrationals, is uncountable.

Exercise 1.10.14 A complex number z is said to be algebraic if there exist $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of algebraic numbers is countable.

Exercise 1.10.15 Prove by induction that,

1.

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}, \ \forall n \in \mathbb{N},$$

2.

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2 \ \forall n \in \mathbb{N},$$

3.

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots + \frac{1}{(3n-2) \cdot (3n+1)} = \frac{n}{3n+1}, \ \forall n \in \mathbb{N},$$

4.

$$n^2 \leq n!, \ \forall n \geq 4,$$

5.

$$2^n > n^2, \ \forall n \geq 5,$$

6.

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \ \forall n \in \mathbb{N}, \ x \in \mathbb{R}, \text{ such that } x \neq 1,$$

Exercise 1.10.16 Prove by induction that if A and B are square matrices such that

$$AB = BA,$$

then

$$AB^n = B^n A, \ \forall n \in \mathbb{N}.$$

Exercise 1.10.17 Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

$$f(n + m) = f(n) + f(m), \ \forall m, n \in \mathbb{N}.$$

1. Prove that

$$f(na) = nf(a), \ \forall n \in \mathbb{N}.$$

   Hint: Fix $a \in \mathbb{N}$ and define

$$A = \{n \in \mathbb{N} \ : \ f(na) = nf(a)\},$$

   and prove by induction that

$$A = \mathbb{N}.$$

2. Prove that there exists $b \in \mathbb{N}$ such that

$$f(n) = bn, \ \forall n \in \mathbb{N}.$$

Exercise 1.10.18 Let $x, y \in \mathbb{R}$ be such that

$$x < y + \varepsilon, \ \forall \varepsilon > 0.$$

Prove formally that

$$x \leq y.$$

Hint: Suppose, to obtain contradiction, that $x > y$.

Exercise 1.10.19 Let $a, b \in \mathbb{R}$ be such that $0 < a < b$. Prove by induction that

$$0 < a^n < b^n, \ \forall n \in \mathbb{N}.$$

Exercise 1.10.20 Let $a \in \mathbb{R}$ be such that $0 < a < 1$.
Let $\varepsilon > 0$. Prove that there exists $n_0 \in \mathbb{N}$ such that

$$0 < a^{n_0} < \varepsilon.$$

Show also that if $n > n_0$, then

$$0 < a^n < a^{n_0} < \varepsilon.$$

Exercise 1.10.21 Let $K \in \mathbb{R}$ be such that $K > 0$. Let $a \in \mathbb{R}$ be such that $a > 1$.
Prove formally that there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$, then

$$a^n > K.$$

Exercise 1.10.22 Let $A, B \subset \mathbb{R}$ be nonempty upper bounded sets. Define

$$A + B = \{x + y \ : \ x \in A \text{ and } y \in B\}.$$

Show that $A + B$ is upper bounded and

$$\sup(A + B) = \sup A + \sup B.$$

Exercise 1.10.23 Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is said to be upper bounded if its range

$$f(X) = \{f(x) \ : \ x \in X\},$$

is upper bounded. In such a case we define the supremum of f on X by

$$\sup f = \sup\{f(x) \ : \ x \in X\}.$$

Given two functions $f, g : X \to \mathbb{R}$, the sum $(f + g) : X \to \mathbb{R}$ is defined by

$$(f + g)(x) = f(x) + g(x), \ \forall x \in X.$$

Prove that if $f, g : X \to \mathbb{R}$ are upper bounded, then so is $f + g$ and also prove that

$$\sup(f + g) \leq \sup f + \sup g.$$

Finally, give an example for which the strict inequality is valid.

Exercise 1.10.24 Let $A \subset \mathbb{R}$ be a nonempty bounded set. Define $-A$ by

$$-A = \{-x \ : \ x \in A\}.$$

Prove that

$$\inf A = -\sup(-A).$$

Exercise 1.10.25 Let $A \subset \mathbb{R}$ be a nonempty bounded set and let $c > 0$. Define cA by

$$cA = \{cx \ : \ x \in A\}.$$

show that

$$\sup(cA) = c \sup A,$$

and

$$\inf(cA) = c \inf A.$$

Exercise 1.10.26 Let $A, B \subset \mathbb{R}^+ = [0, +\infty)$ be nonempty bounded sets. Define

$$A \cdot B = \{xy \ : \ x \in A \text{ and } y \in B\}.$$

Prove that

$$\sup(A \cdot B) = \sup A \ \sup B$$

and

$$\inf(A \cdot B) = \inf A \ \inf B.$$

Exercise 1.10.27 Verify if the sets below indicated are countable or uncountable. Please justify your answers.

1. The set of all sequences having only 0 and 1 entries, with exactly 3 entries equal to 1.

fabio.silva.botelho@gmail.com

2. For $k \in \mathbb{N}$, the set of all sequences having only 0 and 1 entries, with at most k entries equal to 1.
3. $$\mathscr{A} = \{\{a_n\} \ : \ a_n \in \mathbb{N} \cup \{0\}, \ \ \text{such that } a_n = 0, \ \forall n \in \mathbb{N}, \\ \text{except for a finite number of } n's\}.$$
4. $$\mathscr{B} = \{\{a_n\} \ : \ a_n \in \mathbb{N} \text{ and } a_n \geq a_{n+1}, \forall n \in \mathbb{N}\}.$$
5. $$\mathscr{C} = \{\{a_n\} \ : \ a_n \text{ is prime } \forall n \in \mathbb{N}\}.$$
6. $$\mathscr{D} = \{\{a_n\} \ : \ a_n \in \mathbb{N} \text{ and } a_{n+1} \text{ is a multiple of } a_n, \ \forall n \in \mathbb{N}\}.$$
7. $$\mathscr{E} = \{\{a_n\} \ : \ a_n \in \mathbb{N} \text{ and } a_{n+1} \text{ is a divisor of } a_n, \ \forall n \in \mathbb{N}\}.$$
8. The set of all polynomials in x with rational coefficients.
9. The set of all power series $\sum_{n=0}^{\infty} a_n x^n$, such that $a_n \in \mathbb{Z}, \ \forall n \in \mathbb{N} \cup \{0\}$.

Exercise 1.10.28 Prove that if a set B is countable and there exists an injective function

$$f : A \to B,$$

then A is countable.

Exercise 1.10.29 Let A be a countable set and B be a finite set. Constructing a bijection between $\mathbb{N}$ and $A \cup B$, show that $A \cup B$ is countable.