4-33. (*A first course in complex variables.*) If $f\colon \mathbf{C} \to \mathbf{C}$, define f to be ***differentiable*** at $z_0 \in \mathbf{C}$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. (This quotient involves two complex numbers and this definition is completely different from the one in Chapter 2.) If f is differentiable at every point z in an open set A and f' is continuous on A, then f is called ***analytic*** on A.

(a) Show that $f(z) = z$ is analytic and $f(z) = \bar{z}$ is not (where $\overline{x + iy} = x - iy$). Show that the sum, product, and quotient of analytic functions are analytic.

(b) If $f = u + iv$ is analytic on A, show that u and v satisfy the *Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}.$$

Hint: Use the fact that $\lim_{z \to z_0} [f(z) - f(z_0)]/(z - z_0)$ must be the same for $z = z_0 + (x + i \cdot 0)$ and $z = z_0 + (0 + i \cdot y)$ with $x,y \to 0$. (The converse is also true, if u and v are continuously differentiable; this is more difficult to prove.)

(c) Let T: $\mathbf{C} \to \mathbf{C}$ be a linear transformation (where $\mathbf{C}$ is considered as a vector space over $\mathbf{R}$). If the matrix of T with respect to the basis $(1,i)$ is $\begin{pmatrix} a,b \\ c,d \end{pmatrix}$ show that T is multiplication by a complex number if and only if $a = d$ and $b = -c$. Part (b) shows that an analytic function f: $\mathbf{C} \to \mathbf{C}$, considered as a function f: $\mathbf{R}^2 \to \mathbf{R}^2$, has a derivative $Df(z_0)$ which is multiplication by a complex number. What complex number is this?

(d) Define

$$d(\omega + i\eta) = d\omega + i\, d\eta,$$

$$\int_c \omega + i\eta = \int_c \omega + i \int_c \eta,$$

$$(\omega + i\eta) \wedge (\theta + i\lambda) = \omega \wedge \theta - \eta \wedge \lambda + i(\eta \wedge \theta + \omega \wedge \lambda),$$

and

$$dz = dx + i\, dy.$$

Show that $d(f \cdot dz) = 0$ if and only if f satisfies the Cauchy-Riemann equations.

(e) Prove the *Cauchy Integral Theorem:* If f is analytic on A, then $\int_c f\, dz = 0$ for every closed curve c (singular 1-cube with $c(0) = c(1)$) such that $c = \partial c'$ for some 2-chain c' in A.

(f) Show that if $g(z) = 1/z$, then $g \cdot dz$ [or $(1/z)dz$ in classical notation] equals $i\, d\theta + dh$ for some function h: $\mathbf{C} - 0 \to \mathbf{R}$. Conclude that $\int_{c_{R,n}} (1/z)dz = 2\pi i n$.

(g) If f is analytic on $\{z\colon |z| < 1\}$, use the fact that $g(z) = f(z)/z$ is analytic in $\{z\colon 0 < |z| < 1\}$ to show that

$$\int_{c_{R_1,n}} \frac{f(z)}{z}\, dz = \int_{c_{R_2,n}} \frac{f(z)}{z}\, dz$$

if $0 < R_1$, $R_2 < 1$. Use (f) to evaluate $\lim_{R \to 0} \int_{c_{R,n}} f(z)/z\, dz$ and conclude:

Cauchy Integral Formula: If f is analytic on $\{z\colon |z| < 1\}$ and c is a closed curve in $\{z\colon 0 < |z| < 1\}$ with winding number n around 0, then

$$n \cdot f(0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z}\, dz.$$