1) Let $M^m \subseteq \mathbb{R}^k$ and $N^n \subseteq \mathbb{R}^l$ be submanifolds. A C^∞ application $f: M \to N$ is a class C^∞ diffeomorphism if it has C^∞ inverse.

a) Show that every parametrization $\phi: U_0 \to M$ is a diffeomorphism onto its image, when the image is seen a submanifold.

b) Show that $P = \{(x, y, z) \in \mathbb{R}^3 : \ z = x^2 + y^2\}$ is a class C^∞ two-dimensional submanifold of $\mathbb{R}^3$. Use projection in the xy plane to show that P is diffeomorph to $\mathbb{R}^2$.

c) Prove the *inverse function theorem* for submanifolds: if $df_p: T_pM \to T_{f(p)}N$ is a linear isomorphism, there exists open sets $p \in U \subseteq M$ and $f(p) \in V \subseteq N$ such that the restriction $f|_U : U \to V$ is a diffeomorphism. (Suggestion: use parametrizations).

d) Given $0 < r < R$ and $U = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 > 0\}$. We have that 0 is the regular value of the function

$$f: (x, y, z) \in U \longmapsto \left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 - r^2,$$

and therefore $M := f^{-1}(0)$ is a two-dimensional submanifold in $\mathbb{R}^3$. Show that

$$\xi : ((x_1, y_1), (x_2, y_2)) \in S^1 \times S^1 \mapsto ((R + ry_1)x_2 \,, (R + ry_1)y_2 \,, rx_1 \in M$$

is well defined and that it is a diffeomorphism, Finish saying that M is diffeomorph to the T^2 torus.