Mean of a function in $\mathbb{R}^n$

We suppose $A \subset \mathbb{R}^n$ and $f: A \to \mathbb{R}^n$ is a measurable function with respect to a measure μ in a sigma-algebra σ of $\mathbb{R}^n$. We want to understand how to take mean of f over a set A under different conditions on A.

In general, the mean of f over A $\left(E(f)\right)$ is defined to be:

$$E(f) = \frac{\int_A f d\mu}{\mu(A)}$$

1. When A is a finite set, $A = \{a_1, \ldots, a_N\}$, and assuming each singleton set $\{a_i\}$ is measurable in the sigma-algebra σ, this takes the form of:

$$E(f) = \sum_{i=1}^{N} f(a_i) \frac{\mu(\{a_i\})}{\mu(A)}$$

   Hence, the mean of f is the weighted average of its values, where the weights are given by the proportion of mass allocated by μ to each element a_i. In the case that μ gives a constant mass of 1 to every element of A, this translates to the usual average of elements:

$$E(f) = \sum_{i=1}^{N} \frac{f(a_i)}{N}$$

   In the case that A has countable elements and positive measure, we only need to take the sum over all natural numbers:

$$E(f) = \sum_{i=1}^{\infty} f(a_i) \frac{\mu(\{a_i\})}{\mu(A)}$$

2. When A has uncountable elements and positive measure, we take the general definition of the mean and integrate over A. If a measure μ is not specified, we assume it to be Lebesgue measure over the Borel sigma-algebra of $\mathbb{R}^n$.
3.

$$E(f) = \frac{\int_A f d\mu}{\mu(A)}$$

4. One can extend the previous definition to sets of measure zero by considering an open covering of A of measure ϵ and taking limits. We define the upper mean value of f as:

$$\lim_{\epsilon \to 0^+} \sup_{\{O | A \subset O, \mu(O) = \epsilon\}} \frac{\int_O f d\mu}{\epsilon}$$

   where the supremum is taken over all open coverings O of A of measure ϵ. Likewise, the lower mean value of f is defined by taking inf instead of sup:

$$\lim_{\epsilon \to 0^+} \underset{\{O | A \subset O, \mu(O) = \epsilon\}}{inf} \frac{\int_O f d\mu}{\epsilon}$$

   Whenever these two quantities coincide, we define the average value of f as their common value:

$$E(f) \coloneqq \sup_{\{O | A \subset O, \mu(O) = \epsilon\}} \lim_{\epsilon \to 0^+} \frac{\int_O f d\mu}{\epsilon} = \inf_{\{O | A \subset O, \mu(O) = \epsilon\}} \lim_{\epsilon \to 0^+} \frac{\int_O f d\mu}{\epsilon}$$

5. Lastly, we suppose that A has infinite measure. Defined the cumulative distribution function of f as

$$dF: \mathbb{R} \to \mathbb{R},$$
$$dF(y) \coloneqq \mu(\{x \in A | f(x) \leq y\})$$

we can define the average value of f as

$$E(f) \coloneqq \int_{-\infty}^{\infty} y dF(y)$$

where this integral is taken in the sense of Lebesgue-Stieltjes (whenever it exists).

6. Do 'almost all' functions have mean?

We can follow the argument presented in example 3.7 of https://www.ams.org/journals/bull/2005-42-03/S0273-0979-05-01060-8/S0273-0979-05-01060-8.pdf

Because a function can always be represented as $f = f^+ - f^-$ we only consider whether positive functions have a mean value. We consider the case of a set A with finite positive measure. In this context having a mean means having a finite integral, and not being integrable means having an infinite integral.

Take $X \coloneqq L^0(A)$ (measurable functions over A), let P denote the one-dimensional subspace of $L^0(A)$ consisting of constant functions (assuming the Lebesgue measure on A) and let $F \coloneqq L^0(A) \setminus L^1(A)$ (measurable functions over A with no finite integral).

If λ_P denotes the Lebesgue measure over P, for any fixed $f \in F$

$$\lambda_P \left(\left\{ \alpha \in \mathbb{R} \middle| \int_A (f + \alpha) d\mu < \infty \right\} \right) = 0$$

Meaning P is a 1-dimensional probe for F, so F is a 1-prevalent set.