Discrete Logistic Growth

In order to choose one alternative over the other, we study the stability of the equilibrium points for each alternative.

An equilibrium that is not stable is undesirable since small deviations from the chosen strategy could lead to the collapse of the Salmon population, which is to be avoided at all costs.

Alternative 1.

We consider the first strategy: *to allow a fixed number of salmon, h, to be harvested each year before the salmon reproduce.*

Mathematically, we write this as

$$N_{t+1} = N_t + RN_t\left(1 - \frac{N_t}{K}\right) - h$$

where h is the fixed number of salmon that is allowed to be harvested.

We denote by N_e the equilibrium points for this equation and solve for it:

$$N_e = N_e + RN_e\left(1 - \frac{N_e}{K}\right) - h$$

$$\frac{R}{K}N_e^2 - RN_e + h = 0$$

This is a quadratic expression in N_e that has solutions:

$$N_{e_1} = \frac{R - \sqrt{R^2 - 4\left(\frac{R}{K}\right)h}}{2\left(\frac{R}{K}\right)}$$

and

$$N_{e_2} = \frac{R + \sqrt{R^2 - 4\left(\frac{R}{K}\right)h}}{2\left(\frac{R}{K}\right)}$$

Naturally, we must assume that $h \leq \frac{RK}{4}$ for the equilibrium points to even exist. Otherwise, the population would become unbounded, either by growing too large or by simply collapsing.

We now proceed to analyze the stability of each equilibrium point.

In general, for a nonlinear discrete system given by an equation of the form

$$X_{t+1} = f(X_t)$$

an equilibrium point X_e will be stable if and only if[1] $|f'(X_e)| < 1$

In our case,

$$f(N_t) = N_e + RN_e\left(1 - \frac{N_e}{K}\right) - h$$

[1]Leah Edelstein-Keshet, *Mathematical Models in Biology*, Society for Industrial and Applied Mathematics, Part I, Chapter 2, page 42

So that

$$f'(N_t) = 1 + R - 2\frac{R}{K}N_t$$

And

$$f'(N_{e_1}) = 1 + R - 2\frac{R}{K}\left(\frac{R - \sqrt{R^2 - 4\left(\frac{R}{K}\right)h}}{2\left(\frac{R}{K}\right)}\right) = 1 + \sqrt{R^2 - 4\left(\frac{R}{K}\right)h} > 1$$

So N_{e_1} is an unstable and, hence, unsustainable equilibrium point. On the other hand,

$$f'(N_{e_1}) = 1 + R - 2\frac{R}{K}\left(\frac{R + \sqrt{R^2 - 4\left(\frac{R}{K}\right)h}}{2\left(\frac{R}{K}\right)}\right) = 1 - \sqrt{R^2 - 4\left(\frac{R}{K}\right)h} < 1$$

N_{e_1} will be stable if and only if

$$-1 < 1 - \sqrt{R^2 - 4\left(\frac{R}{K}\right)h}$$

which is equivalent to

$$h > \frac{R^2 - 4}{4\left(\frac{R}{K}\right)}$$

If $h = \frac{RK}{4}$, giving only one solution to the quadratic equation, $N_e = \frac{K}{2}$, $f'(N_e) = 1$, which means it is unstable.

Therefore, this alternative has a long-term equilibrium that is stable if and only if

$$\frac{R^2 - 4}{4\left(\frac{R}{K}\right)} < h < \frac{RK}{4}$$

Under these two conditions on h we can guarantee stability and, hence, sustainability of the proposed alternative.

To maximize the quantity of salmon that is allowed to be harvested under alternative 1, one would consider $h \approx \frac{RK}{4}$.

Alternative 2.

We consider now the second strategy: *to allow a fixed proportion of the salmon population, H, to be harvested each year before the salmon reproduce.*

Mathematically, we write this as

$$N_{t+1} = N_t + RN_t\left(1 - \frac{N_t}{K}\right) - HN_t$$

where H is the fixed proportion of salmon that is allowed to be harvested.

We denote by N_e the equilibrium points for this last equation and solve for it:

$$N_e = N_e + RN_e\left(1 - \frac{N_e}{K}\right) - HN_e$$

$$\frac{R}{K}N_e^2 + (H - R)N_e = 0$$

This is a quadratic expression in N_e that has two solutions:

$$N_e = 0 \text{ and } N_e = \frac{R-H}{\left(\frac{R}{K}\right)}$$

The first solution is obviously rejected, and the second solution is only feasible if $H < R$

As for the stability, for this dynamic we would have

$$f(N_t) = N_e + RN_e\left(1 - \frac{N_e}{K}\right) - HN_t$$

$$f'(N_t) = 1 + R - 2\frac{R}{K}N_t - H$$

$$f'(N_e) = 1 + R - 2\frac{R}{K}\left(\frac{R-H}{\left(\frac{R}{K}\right)}\right) - H = 1 - R + H$$

This is a stable solution if and only if

$$-1 < 1 - R + H < 1$$

which is equivalent to

$$R - 2 < H < R$$

To maximize the quantity of salmon that is allowed to be harvested under alternative 2, one would consider $H \approx R - 2$.

This would mean that, under equilibrium, the maximum number of salmon allowed to be harvested under alternative 2 would be

$$HN_e \approx (R-2)\frac{R-(R-2)}{\left(\frac{R}{K}\right)} = \frac{2K(R-2)}{R}$$

Conclusion:

Stable equilibrium points exist under both alternatives under certain conditions. This guarantees that the proposed solutions are sustainable in the long term.

The alternative that allows to harvest the largest number of salmons in the long run should be preferred.

Hence, alternative 1 should be preferred over alternative 2 if $\frac{RK}{4} > \frac{2K(R-2)}{R}$, i.e, if $(R-4)^2 > 0$

Which means that as long as $R \neq 4$, alternative 1 should be preferred (so long as the initial conditions are near to the stable equilibrium point).