Vector Space

$V = F[x]$ is the set of all polynomials over a field F and W is the set of these polynomials that have degree less than or equal to n.

V/W is defined as the quotient space of classes $[x] = \{x + w | w \in W\}$, with addition and scalar multiplication defined by taking representatives: $[x] + [y] = [x + y]$, $\alpha[x] = [\alpha x]$

We want to find a basis for V/W.

Let $B = \{x^{n+1}, x^{n+2}, x^{n+3}, \dots\}$. This is an infinite set and will serve as basis for V/W.

To see this, we need to show that any element in V/W can be written as a **finite** linear combination of elements in B, and that any **finite** linear combination of elements in B that is zero has all coefficients as zero.

Let $p(x) \in V/W$. Since $p(x)$ is a polynomial, it will be a finite combination of powers of x. Let $n + m$ be the highest power on $p(x)$ for some $m \in \mathbb{N}$. We can write $p(x)$ as:

$$p(x) = a_1 x^{n+1} + \cdots \ a_m x^{n+m}$$

for some coefficients $a_i \in F, i \leq m$ (with some coefficients being possibly zero).

This is clearly a finite linear combination of elements in B, so B spans V/W

Now, suppose that a finite combination of elements in B equals 0. Since this is a finite combination, let Let $n + m$ be their highest power for some $m \in \mathbb{N}$. Then we have that

$$a_1 x^{n+1} + \cdots \ a_m x^{n+m} = 0$$

Now, two polynomials are equal as algebraic elements if and only if their respective coefficients are zero. This allows us to conclude that the coefficients on the left have to zero, so the set B is linearly independent.

Since V/W has basis set with infinite elements, it is an infinite dimensional space.