Question: Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(2x) + 2f(y) = f(x + y)$ for all $x, y \in \mathbb{Z}$.

Plugging in $x \mapsto (x - 1)$ and $y \mapsto (y + 1)$, we get

$$f(2(x-1)) + 2f(y+1) = f(f(x+y)),$$

and thus in combination with

$$f(2x) + 2f(y) = f(f(x+y)),$$

we find that

$$f(2(x-1)) + 2f(y+1) = f(f(x+y)) = f(2x) + 2f(y).$$

Not rearranging the furthest left side and furthest right side, we obtain

$$f(2(x-1)) - f(2x) = 2f(y) - 2f(y+1).$$

Setting $x = 0$, and dividing through by 2, we see that

$$\frac{f(-2) - f(0)}{2} = f(y) - f(y+1). \tag{1}$$

Writing $m = -\left(\frac{f(-2)-f(0)}{2}\right)$, we find that

$$f(0) - f(1) = -m \Rightarrow f(1) = f(0) + m,$$

and

$$f(1) - f(2) = -m \Rightarrow f(2) = f(1) + m = f(0) + 2m.$$

Now, we will prove that $f(y) = my + b$, where $b = f(0)$, for all $y \in \mathbb{Z}$ by induction. First, given $k \geq 0$, we assume

$$f(k) = mk + b.$$

By (1) we find

$$-m = f(k) - f(k+1) \tag{2}$$

$$\Rightarrow f(k+1) = f(k) + m = km + b + m = m(k+1) + b. \tag{3}$$

Thus by induction $f(y) = mk + b$ for all $y \in \mathbb{Z}$ such that $y \geq 0$. By an entirely similar argument, inducting on $f(-k)$ for $k \geq 0$, we're able to conclude that $f(y) = my + b$ holds for all $y \in \mathbb{Z}$.

Therefore we see that f is a linear equation. Now we substitute this back into $f(2x) + 2f(y) = f(f(x+y))$ to determine possible values of the constants m and b. Thus

$$f(2x) + 2f(y) = f(f(x+y)) \tag{4}$$

$$m(2x) + b + 2(my + b) = f(m(x+y) + b) \tag{5}$$

$$2mx + 2my + 3b = m(mx + my + b) + b \tag{6}$$

$$2(mx + my) + 2b = m(mx + my + b). \tag{7}$$

Taking $x = y = 0$, we obtain

$$2b = mb. \tag{8}$$

Thus we have two separate cases,

1) If $b = 0$, then from (7)

$$2(mx + my) = m(mx + my) \tag{9}$$
$$\Rightarrow (2m - m^2)x = (m^2 - 2m)y \tag{10}$$

Now if $(m^2 - 2m) \neq 0$, we divide through to obtain

$$-x = y, \tag{11}$$

but this is supposed to hold for all $x, y \in \mathbb{Z}$, which is clearly impossible. Thus, it must be the case that $m^2 - 2m = 0 \Rightarrow m(m-2) = 0$. Therefore, we have two possibilities, either $m = 0$, or $m = 2$. Now since we're assuming $b = 0$, if $m = 0$ then f is simply the function sending everything to 0, and indeed

$$f(2x) + 2f(y) = f(f(x + y)) \tag{12}$$
$$0 + 2 \cdot 0 = f(0) \tag{13}$$
$$0 = 0, \tag{14}$$

is satisfied.
Otherwise, plugging $m = 2$ into (10), the equation is in fact satisfied by all $x, y \in \mathbb{Z}$, and therefore $f(y) = 2y$ is another valid solution.

Case 2) If $b \neq 0$, then dividing both sides of (8) by b, we still conclude that $m = 2$, and putting this into (7), we get

$$2(2x + 2y) + 2b = 2(2x + 2y + b) \tag{15}$$
$$4x + 4y + 2b = 4x + 4y + 2b. \tag{16}$$

Thus we find that the equation is satisfied for *any* choice of integer $b \in \mathbb{Z}$. Note too, that b *must* be an integer, for otherwise there would exist some $y \in \mathbb{Z}$ such that $f(y) = 2y + b$ is not an integer.

Therefore, the equations $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(2x) + 2f(y) = f(f(x + y))$ are precisely those of the form $f(y) = 2y + b$ for any $b \in \mathbb{Z}$, or the function $f(y) = 0$ for all $y \in \mathbb{Z}$.